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HYDRAULIC VARIABLE-SPEED TRANSMISSIONS AS SERVOMOTORS.

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ABSTRACT.

The hydraulic transmission is an important form of servomotor about which little has been written from the viewpoint of the servomechanisms specialist. This paper presents an analysis of the hydraulic transmission; and, for three transmissions, a comparison between the predicted behavior based on the analysis and the measured behavior observed by frequency response tests. Discrepancies between predicted and measured performance are discussed and conclusions drawn. Suggestions are made for improvement of transmission performance. Data necessary for calculations on the transmissions tested are given in the appendix.

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TRANSMISSION THEORY.

1. Servomechanisms and Hydraulic Transmissions.

Hydraulic transmissions are widely used as power elements, or servomotors, in a variety of servomechanisms and automatic control systems. For many military applications hydraulic transmissions constitute the only feasible servomotors because of their light weight. But even where weight is of no great importance, hydraulic transmissions are frequently chosen because of performance considerations. Some examples of applications of hydraulic transmissions are:

1. Gun positioning apparatus.
2. Ship steering gear.
3. Antenna positioning equipment.
4. Tensioning drives in metal, paper, and textile industries.
5. Machine tools.

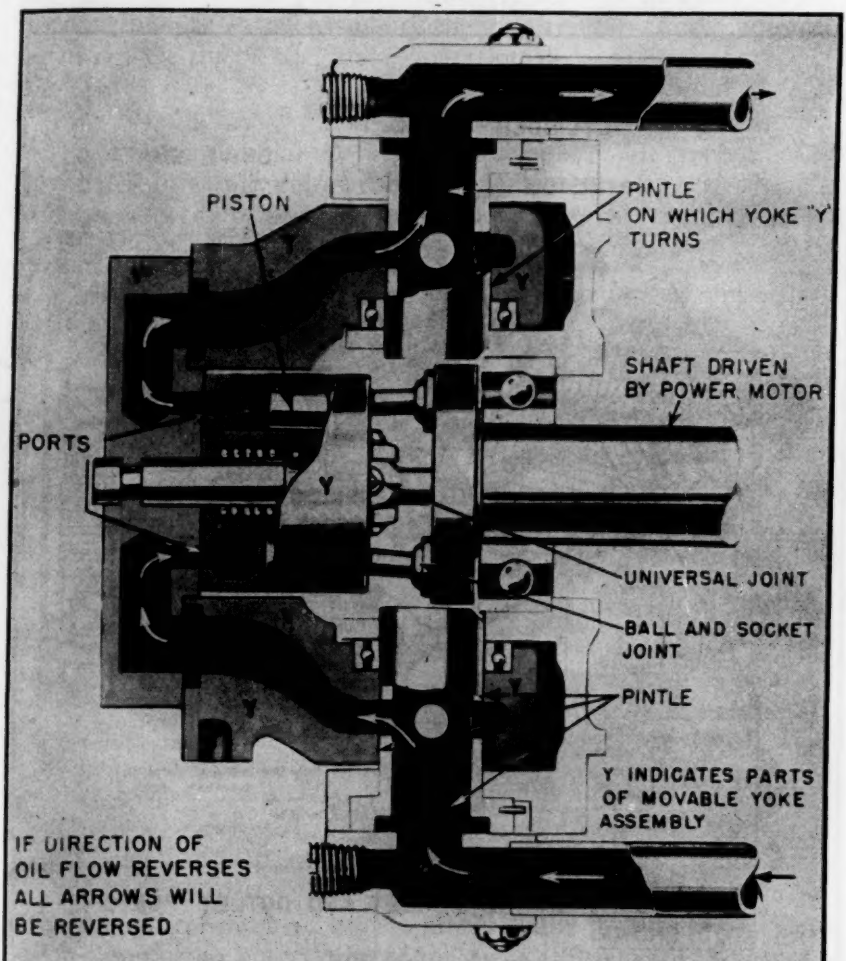
In order intelligently to design and apply hydraulic transmissions as servomotors, it is necessary to have some conception of the factors that affect their performance. This can be done by analyzing their operation from the point of view of the servomechanisms engineer.

Any element in a servomechanism that comes under the scrutiny of a servomechanist is first analyzed or tested to determine its dynamic behavior. In analysis he attempts to write differential equations which will express its operation; in testing he either takes oscillograms of its behavior for various types of input disturbances or measures the output motion when its input is oscillated at various frequencies and amplitudes. If possible he prefers both to analyze and test a piece of equipment in order to check theory against experiment. Before applying the method of the servomechanist to the hydraulic variable-speed transmission a brief description of a typical apparatus will be given.

2. Description of Hydraulic Transmission Components.

Figure 1 shows pictorially a variable displacement pump. In the transmission, this element converts the mechanical energy of input shaft rotation into hydraulic energy of fluid flow. The type of pump shown in Fig. 1 is manufactured by Vickers, Inc. of Detroit, Michigan. When the cylinder block axis is tilted with respect to the shaft axis, the pistons are caused to move back and forth in the cylinders as the whole assembly revolves. The holes at the end of the cylinders commutate with ports in the valve plate so arranged that all cylinders exhausting oil are communicating with one port and all cylinders taking in oil are communicating with the other, oppositely located, port. Passages lead from the ports through the tilting yoke and bearing pintles to the outside pipe connections. The amount of oil per revolution taken in one connection and pumped out the other depends upon the angle tilt of the yoke and

cylinder block. If the angle is small, the back and forth excursion of pistons is small and a small volume of oil is pumped; whereas, if the angle is large, the excursion is large and a large volume is pumped. At zero

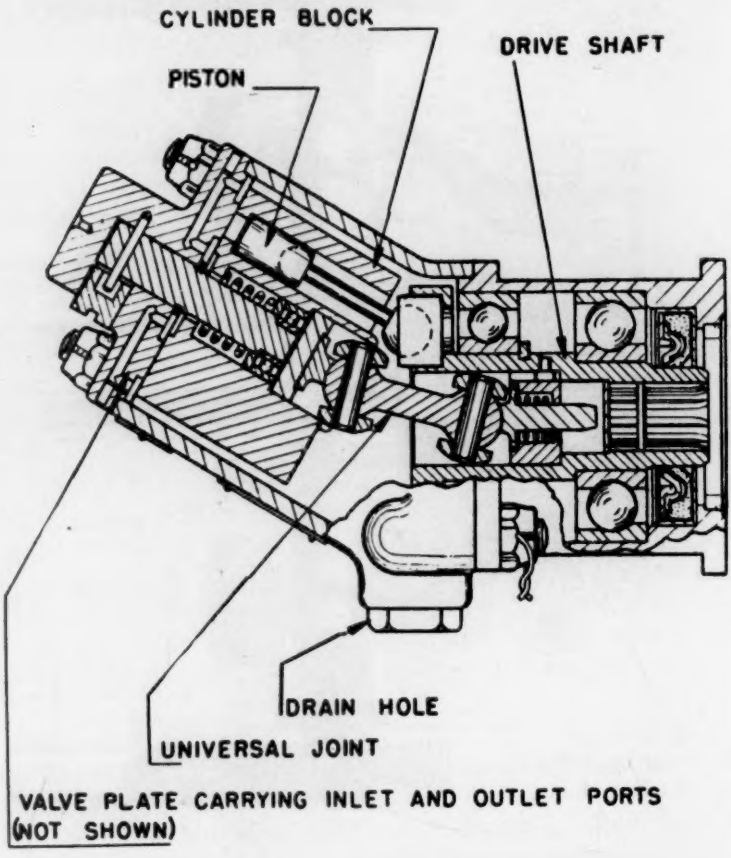


SECTION THROUGH HYDRAULIC PUMP
MOVABLE YOKE SHOWN IN DARKER COLOR SEE Y,Y,Y
FIGURE 1

tilt angle the excursion is zero and no oil is pumped. If the angle is made negative reversed flow will occur.

Figure 2 shows a constant displacement hydraulic motor. In construction it is similar to the pump except the angle of tilt is fixed and no tilting yoke is necessary. Oil pressure at either of the two main con-

nections is transmitted to the communicating pistons. The resultant piston forces are transmitted by the rods to the drive socket ring. Because of the tilt angle these forces will tend to rotate the socket ring if the oil pressures from which they originate are not equal. In other



SECTION THROUGH HYDRAULIC MOTOR

FIGURE 2

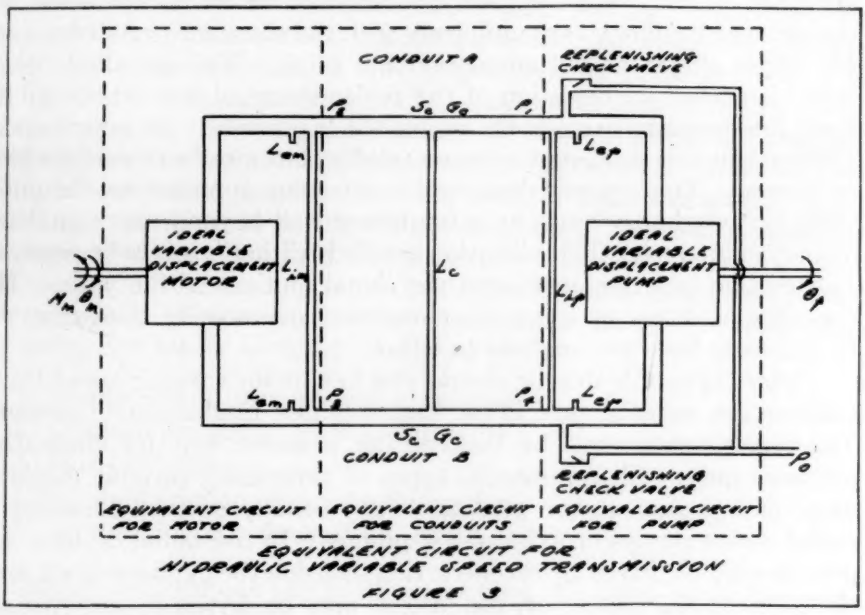
words a moment will be produced on the output shaft by unbalanced pressures at the main connections.

A hydraulic variable speed transmission may be formed out of a variable displacement pump and a constant displacement motor by con-

necting the outlet-inlet of the pump to inlet-outlet connections of the motor and filling the assembly with oil. If the pump shaft is rotated at constant speed, the motor will tend to turn at a speed dependent upon the tilt angle of the pump. However, leakage of fluid through the piston-cylinder clearances of both the pump and motor would soon cause air to replace the oil and make operation very unsatisfactory.

3. Replenishment, Relief Valves, and Dither.

To prevent the mal-operation that would result from leakage, it is customary to supply oil at constant pressure from an auxiliary source to the low-pressure pump connection. Since this connection changes



EQUIVALENT CIRCUIT FOR HYDRAULIC VARIABLE SPEED TRANSMISSION

FIGURE 3

with reversals of moment on the motor shaft, it is necessary to supply replenishing oil to both pump connections through check valves which prevent high pressure oil caused by motor shaft moments from entering the replenishing system. (See Fig. 3.)

The oil that leaks through the pistons and other clearances is collected in the case of the unit in question and returned to the replenishing sump by drain lines. Thus the motor and pump generally have, besides the two main connections for each, a drain connection. Sometimes rather than using a separate sump, the pump case is used as a sump, and occasionally, the pump and motor are fitted with a single case which acts as a sump and avoids the need for drain lines and main conduits. The replenishing pressure source is usually a gear pump driven

from the same shaft as the main pump and fitted with a regulating valve or relief valve to maintain constant pressure. This pressure must be high enough not only to prevent air entrainment but also to prevent cavitation.

The pressure across the motor and the moment developed by the motor are directly related. To prevent excessive output torque or moments it is necessary to provide pressure limiting means. These comprise a pair of relief valves inserted across the pump connections or main conduits and arranged to bypass oil from one side to the other whenever the differential (moment producing) pressure exceeds preset limits.

The combination of hydraulic pump, motor and replenishing means would seem to operate as an excellent variable speed transmission. However, if there is an appreciable moment on the motor shaft, the output speed will not vary uniformly with the tilt angle. At very small tilt angles almost no output motion will occur. This so-called "dead-spot" is caused by cessation of the replenishing oil flow whenever the main line pressure exceeds the replenishing pressure. In nonreversing applications this dead-spot gives no trouble, but most servomotors have to reverse. To overcome dead-spot in reversing applications, the pump yoke is caused to vibrate at a frequency well beyond any signal frequency of interest. This vibration is called "dither"; it is to be regarded as a motion superimposed upon the signal motions of the yoke. The necessity of dither or some other means of overcoming dead-spot will be apparent from the analysis to follow.

Other types of hydraulic pumps and motors for variable-speed transmissions are manufactured in the United States, England and Germany. The type manufactured by Vickers, Inc. is shown here for illustrative purposes only. All commercial types of reversible, variable displacement pumps seem to use pistons. They usually may be classified as radial piston pumps or axial piston pumps. In the radial designs, the pistons may be driven by eccentric rings outside the cylinder block with the ports at the center; or, the pistons may be driven by eccentric or crank arrangements at the center with the ports located near the axis on either side. The axial piston pumps are either similar to the design shown here where the cylinder block tilts, or a form where the socket ring tilts. Modern hydraulic pumps and motors represent some of the finest examples of the machinery builders' art. Tolerances are necessarily held very close; the piston-cylinder clearance, for example, may be held as low as three ten-thousandths of an inch.

4. Analysis of the Hydraulic Transmission.

The analysis presented here is applicable within the limits of its assumptions to any type of hydraulic transmission discussed above. However, as in any analysis, idealizations are made and the conclusions

based on the analysis will be found erroneous to the extent that the grossness of idealization exceeds the precision of observation.

Figure 3 shows an "equivalent circuit" of a hydraulic transmission. (Relief valves have been omitted as the analysis will apply only to normal operation.) It is assumed that both the pump and motor have an infinite number of pistons so that in each the oil displaced is proportional to the shaft rotation. (In actual equipment the oil displaced may be considered proportional to the shaft rotation with a small ripple superimposed; the designer should always take precautions to minimize this ripple.) It is further assumed that an actual pump or motor acts like a pump or motor without leakage, but which has leaks at its connections. The leaks are assumed to be of two types; one, leaks between the line and case which are called direct leaks, and two, leaks between line and line which are called differential leaks. All leaks are assumed to have the characteristic of flow proportional to pressure difference. In Fig. 3, the leakage coefficients L_{ep}, L_{em} are written near the direct pump and motor leaks; L_{ip} and L_{im} near the differential leaks. Leakage past the pistons of an actual pump or motor correspond to contributions to the direct leaks. Leakage between ports at the valve plates correspond to contributions to the differential leaks. Additional assumptions are that the pump and motor parts are perfectly rigid and that the friction and inertia effects of the fluid within the motor and pump are negligible. A summary of the approximations used in analyzing the pump and motor so far is:

1. Displacement proportional to rotation.
2. Leakage lumped at connections.
3. Perfect rigidity.
4. No fluid friction effects inside units.
5. No fluid inertia effects inside units.

Similar assumptions are made concerning the behavior of the conduits connecting the pump and motor. The two conduits are assumed to be of the same length; to be perfectly rigid; to impose two component pressure drops, one proportional to flow, the other proportional to rate of change of flow. In addition it is assumed that a small differential leak exists at the center of the conduits. (Such a leak may be added on purpose or it may exist because of valve leakage if valves are placed in the lines for special purposes.)

In operation, when the motor is loaded, one conduit will be at substantially replenishing pressure P_0 and the other at some higher pressure depending upon the load. The relationship of motor and pump rotation will be found by writing the equation of continuity for the high pressure side, that is, the fluid flow out of the pump must equal the fluid flow into the motor less losses and shrinkages. In writing this equation, it will

be assumed that when the pump shaft velocity, $\dot{\theta}_p$, is positive and the pump displacement per radian (determined by its tilt angle), D_p, is positive, fluid flows from the pump into conduit A and causes the motor to turn with a positive velocity, $\dot{\theta}_m$. The motor load torque, M_m, will be assumed to be positive when it tends to reduce $\dot{\theta}_m$. A positive M_m thus causes high pressure in conduit A. Within the limits of our assumptions concerning the motor, the pressure drop across it, $P_2 - P_3$, is given by M_m/D_m where D_m is the motor displacement per radian and is a positive constant. (This may be easily shown by equating the mechanical and hydraulic work.)

To compute the fluid losses and shrinkages we will make these additional assumptions:

1. For purposes of computing the conduit pressure drops, the flow in the conduits will be assumed equal to the flow of oil into the motor. Second order effects caused by leakage at various points and shrinkage throughout the high pressure oil will be neglected.
2. For purposes of computing the shrinkage of fluid volume caused by changing pressure, a constant compressibility, C_h, will be assumed.
3. The pressure drop across the mid-conduit leak will be assumed to be the mean of the pressure differentials at the conduit extremities.

It is necessary to compute the pressures P_1, P_2, P_3, P_4 (see Fig. 3) at the conduit extremities before the losses and shrinkage can be computed. Under the condition of high pressure in conduit A, $P_4 = P_0$, the replenishing pressure. $P_4 - P_3$ can be computed from the motor flow and line drop coefficients, $P_2 - P_3$ from the motor load; and $P_1 - P_2$ equals $P_3 - P_4$. Using the notation defined in the table of symbols at the end of this paper and referring to Fig. 3, the pressure equations are written as follows:

$$P_4 = P_0, \tag{1}$$

$$P_3 = P_0 + G_c D_m \dot{\theta}_m + S_c D_m \ddot{\theta}_m, \tag{2}$$

$$P_2 = P_0 + G_c D_m \dot{\theta}_m + S_c D_m \ddot{\theta}_m + \frac{M_m}{D_m}, \tag{3}$$

$$P_1 = P_0 + 2G_c D_m \dot{\theta}_m + 2S_c D_m \ddot{\theta}_m + \frac{M_m}{D_m}. \tag{4}$$

The loss from the high pressure side caused by a leak is equal to the pressure drop across the leak times the leakage coefficient. The shrinkage (rate of change of volume) is equal to the product of volume of fluid, compressibility and rate of change of pressure. There is a third type of loss caused by commutation which may be termed commutation loss.

Fluid at low pressure in a pump cylinder is brought into contact with high pressure oil before it is compressed by piston movement. Some of the high pressure oil displaced from the other cylinders must be used to compress the oil in the cylinder undergoing commutation. This loss is the product of the compressibility, the volume of oil in the cylinder undergoing commutation, the pressure difference and the commutation rate in number of cylinders per second. Because of our assumption of an infinite number of cylinders, the commutation loss is proportioned to speed of rotation. (In an actual pump or motor the loss varies with position and gives rise to pulsations observed as noise. Special care should be taken in port design to minimize the noise developed from commutation.) In expressing the commutation loss, use will be made of the commutation loss coefficients, K_p for the pump and K_m for the motor. The loss, in case of the pump for example, is $K_p\dot{\theta}_p(P_1 - P_4)$ and K_p varies with the displacement or tilt angle. Only the unit acting as a pump gives rise to a commutation loss which affects the continuity equation of interest. The unit which is motoring produces an opposite effect in the low pressure side, but that is of no importance except in so far as it affects efficiency.

For the hydraulic transmission of Fig. 3, the equation of continuity (under the condition of the pump driving the motor) is:

Free flow of pump	$D_p\dot{\theta}_p =$
Free flow of motor	$+ D_m\dot{\theta}_m$
Direct leakage of pump	$+ L_{ep}(P_1 - P_s)$
Differential leakage of pump	$+ L_{ip}(P_1 - P_4)$
Differential leakage of conduit	$+ L_{c\frac{1}{2}}[(P_1 + P_2) - (P_3 + P_4)]$
Direct leakage of motor	$+ L_{em}(P_2 - P_s)$
Differential leakage of motor	$+ L_{im}(P_2 - P_3)$
Shrinkage in conduit	$+ V_eC_h\frac{1}{2}(\dot{P}_1 + \dot{P}_2)$
Shrinkage in pump	$+ V_pC_h\dot{P}_1$
Shrinkage in motor	$+ V_mC_h\dot{P}_2$
Commutation loss of pump	$+ K_p\dot{\theta}_p(P_1 - P_4).$ (5)

P_s is the sump pressure into which all direct leaks drain; the other symbols are as defined in the Table of Symbols located in the Appendix.

Substituting in the values of pressure from equations (1) through (4) and rearranging, permits expressing the equation of continuity as:

$$\begin{aligned}\frac{D_p}{D_m}\dot{\theta}_p = \dot{\theta}_m &+ A_1\dot{\theta}_m + A_2\ddot{\theta}_m + A_3\dddot{\theta}_m \\ &+ B_1M_m + B_2\dot{M}_m \\ &+ C_1\dot{\theta}_p\dot{\theta}_m + C_2\dot{\theta}_p\ddot{\theta}_m \\ &+ D_1\dot{\theta}_pM_m \\ &+ F_4(P_0 - P_s) \qquad (6)\end{aligned}$$

where:

$$A_1 = G_c(2L_{ep} + 2L_{ip} + L_c + L_{cm}), \tag{7}$$

$$A_2 = S_c(2L_{ep} + 2L_{ip} + L_c + L_{cm}) + G_cC_h(2V_p + 1\tfrac{1}{2}V_c + V_m), \tag{8}$$

$$A_3 = S_cC_h(2V_p + 1\tfrac{1}{2}V_c + V_m), \tag{9}$$

$$B_1 = \frac{L_{ep} + L_{ip} + L_c + L_{cm} + L_{im}}{D_m^2} = \frac{L_t}{D_m^2}, \tag{10}$$

$$B_2 = \frac{C_h(V_p + V_c + V_m)}{D_m^2} = C_h\frac{V_t}{D_m^2}, \tag{11}$$

$$C_1 = 2G_cK_p, \tag{12}$$

$$C_2 = 2S_cK_p, \tag{13}$$

$$D_1 = \frac{K_p}{D_m^2}, \tag{14}$$

$$F_4 = \frac{L_{ep} + L_{cm}}{D_m}. \tag{15}$$

Equation (6) is the fundamental equation governing the output shaft motion when energy is being supplied to the load. In deriving this equation, friction and inertia loading of the motor has been lumped with the external load. When energy is being removed from the load a similar equation applies except the commutation loss occurs in the motor. The equation for the condition of energy being removed from the load (braking) is:[1]

$$\begin{aligned}\frac{D_p}{D_m}\dot{\theta}_p = {} & \dot{\theta}_m + A_1\ddot{\theta}_m + A_2\dddot{\theta}_m + A_3\ddddot{\theta}_m \\ & + B_1M_m + B_2\dot{M}_m \\ & + E_1\dot{\theta}_mM_m \\ & - F_4(P_0 - P_s)\end{aligned} \tag{16}$$

where A_1, A_2, A_3, B_1, B_2, F_4 are defined above, and

$$E_1 = K_m/D_m^2. \tag{17}$$

Equations (6) and (16) together define the output shaft motion. The dead-spot mentioned previously originates from the change of sign of the $F_4(P_0 - P_s)$ term when the condition of the transmission changes from that of supplying energy to the load to that of absorbing energy from the load. If the output shaft is ever so slightly loaded, the input, $(D_p/D_m)\theta_p$, must be made equal to $\pm F_4(P_0 - P_s)$ before motion can

[1] It is possible, upon sudden diminuation of the motor moment, M_m, for the pressure in both conduits to rise above the replenishing pressure because of the resultant expansion of the fluid. This transient condition is not accounted for in the analysis presented here.

occur. In other words, our equations predict that the pump must have a finite stroke or tilt angle before motion can occur. Actually this is not strictly true because the replenishing check valves do not have a perfectly sharp characteristic as implicitly assumed in this analysis. Rather than an absolute dead spot there usually is a region where the transmission is "soft." An amplitude of dither sufficient to cover up this region of softness is ordinarily superimposed upon the stroke called for by the signal.

Unless the relationship of θ_m and M_m is known, analysis must stop at equations (6) and (16). However, some loads may be considered to be composed of pure inertia and pure viscous friction. Using the symbols J_l, J_m and J_0 to represent the load, motor, and total inertia respectively; F_l, F_m and F_0 to represent the load, motor, and total damping respectively, the load moment may be expressed as (using the symbol p for operator d/dt):

$$M_m = (J_0 p^2 + F_0 p)\theta_m. \tag{18}$$

If the effect of dither is assumed to be merely the removal of the $\pm F_4(P_0 - P_s)$ term in equations (6) and (16), and if the net effect of the terms involving C_1, C_2, D_1, E_1 can be assumed to be equivalent to an increase in the pump leakage, then equations (6) and (16) may be combined into a single approximate equation:

$$\begin{aligned}\frac{D_p}{D_m}\dot{\theta}_p = {} & \dot{\theta}_m + A_1'\dot{\theta}_m + A_2'\ddot{\theta}_m + A_3\dddot{\theta}_m \\ & + B_1 M_m + B_2 \dot{M}_m \\ & + G_1 M_m\end{aligned} \tag{19}$$

where A_1' and A_2' are A_1 and A_2 as before defined with compensation added for removal of C_1 and C_2; G_1 is some arbitrary compromise between $D_1\dot{\theta}_p$ and $E_1\dot{\theta}_m$. In this equation D_p, $\dot{\theta}_p$, M_m and $\dot{\theta}_m$ symbolize the signal frequency components, or the instantaneous values of the variables with dither components removed, rather than the true instantaneous values as in equations (6) and (16). Since an "ideal" transmission would have $\dot{\theta}_m$ equal to $(D_p/D_m)\dot{\theta}_p$, the quantity $(D_p/D_m)\dot{\theta}_p$ will be defined as $\dot{\theta}_{mi}$, the output velocity of an ideal transmission. Equation (19) can be rewritten in terms of θ_{mi} as:

$$\begin{aligned}\theta_{mi} = {} & (A_3 p^2 + A_2' p + A_1' + 1)\theta_m \\ & + \left(B_2 + \frac{B_1 + G_1}{p}\right) M_m.\end{aligned} \tag{20}$$

Substitution of the value of M_m given by equation (18) yields:

$$\begin{aligned}\theta_{mi} = {} & \{(A_3 + B_2 J_0)p^2 + [(B_1 + G_1)J_0 + B_2 F_0 + A_2']p \\ & + [(B_1 + G_1)F_0 + A_1' + 1]\}\theta_m.\end{aligned} \tag{21}$$

It is sometimes more convenient to write this equation in the form:

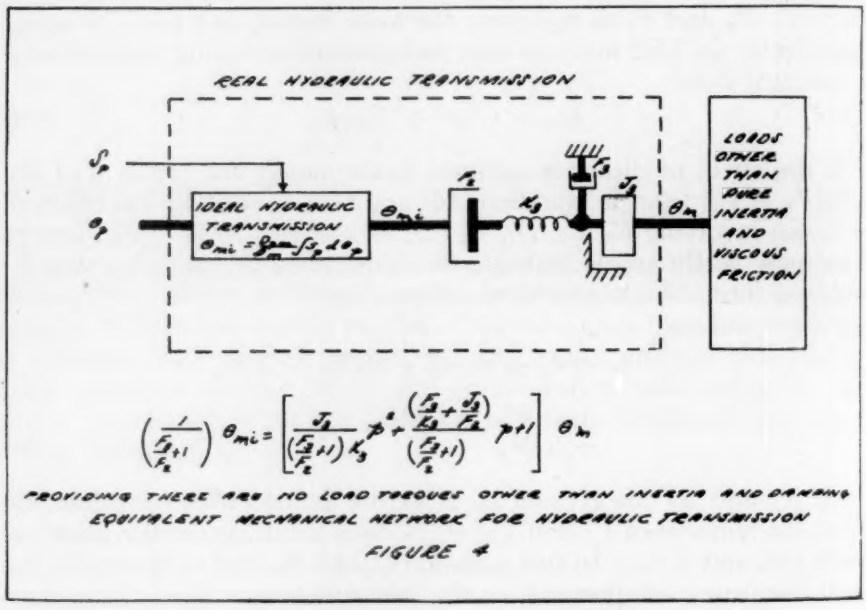
$$S_1\theta_{mi} = \left[\left(\frac{T_1}{2\pi}\right)^2 p^2 + Z_1\frac{T_1}{\pi}p + 1\right]\theta_m \tag{22}$$

where:

$$S_1 = \frac{1}{(B_1 + G_1)F_0 + A_1' + 1} \tag{23}$$

$$T_1 = 2\pi\sqrt{(A_3 + B_2J_0)S_1} \tag{24}$$

$$Z_1 = \pi\frac{[(B_1 + G_1)J_0 + B_2F_0 + A_2']}{T_1}S_1. \tag{25}$$


REAL HYDRAULIC TRANSMISSION
IDEAL HYDRAULIC TRANSMISSION
LOADS OTHER THAN PURE INERTIA AND VISCOUS FRICTION
PROVIDING THERE ARE NO LOAD TORQUES OTHER THAN INERTIA AND DAMPING
EQUIVALENT MECHANICAL NETWORK FOR HYDRAULIC TRANSMISSION
FIGURE 4

5. Equivalent Network for Hydraulic Transmissions.

By making some rather gross approximations it has been possible to arrive at a simple linear differential equation as an expression of the behavior of a hydraulic variable-speed transmission. This equation suggests as an equivalent for a real hydraulic transmission the mechanical network shown in Fig. 4. Although the equation shown below the diagram is valid only for pure inertia and viscous loading, the equivalent network may be used with any kind of load. Obviously, electrical networks analogous to the hydraulic transmission may also be derived.

6. Theoretical Response to Sinusoidal Inputs.

Some servomechanists study servomotors and other components of servomechanisms in terms of frequency response characteristics similar

to those often utilized in connection with electrical networks and wave filters. The fundamental basis of this approach is the mathematical fact that *any system for which a linear differential equation with constant coefficients can be written has a behavior with respect to sinusoidal input signals that characterizes that system* to the same degree that the differential equation does. Furthermore, this characteristic behavior pattern is readily determined by experiment.

The frequency response method is applicable to any linear system for which an input variable and an output variable can be defined so that the output variable (as a function of time) is functionally related to the input variable. It is then found that the ratio of output amplitude to input amplitude, when the input is oscillated sinusoidally at a frequency, f, has a definite value independent of the input (or output) amplitude. If the frequency, f, is changed to a new value, a new ratio will generally exist. In this way, by letting f take on many values, the amplitude ratio as a function of frequency can be found. In a similar manner, the phase angle of an output relative to the input as a function of frequency can be found. The amplitude and phase as functions of frequency together constitute the frequency response characteristics of the system. It is to be emphasized that this is a *steady-state* frequency response, i.e., for each value of frequency the amplitude ratio and phase angle of the characteristic functions apply only if the input has been kept at this frequency and at a constant amplitude long enough for transient effects to have become negligible.

For the hydraulic transmission the input may be considered to be θ_{mi}. In actual use, of course, the stroke of the pump would be the physical input. (The stroke, S_p, is defined for purposes of this paper as the fraction of maximum displacement at which the pump instantaneously operates. Defining D_{pm} as the maximum pump displacement, SD_{pm} is equal to D_p. Stroke is defined in this way to make it dimensionless and therefore applicable to all types of pumps.) If the pump is operating at constant speed, the stroke is proportional to θ_{mi}; in practice, the pumps incorporated into hydraulic transmissions used as servomotors operate at substantially constant speed. The output of the transmission will be considered to be θ_m.

It is a simple mathematical transformation to go from the differential equation (22) to an equation involving complex variables from which the steady-state frequency response may be predicted. This transformation amounts to substitution of $j2\pi f$ for p where $j = \sqrt{-1}$. Equation (26) below is equation (22) so transformed. θ_{mi} and θ_m are now complex quantities related to, but not the θ_m and θ_{mi} of, the differential equation:

$$S_1\theta_{mi} = [j2Z_1T_1f + 1 - (T_1f)^2]\theta_m. \qquad (26)$$

The complex ratio of θ_m to θ_{mi} is often written as a magnitude and

angle. The magnitude is equal to the amplitude ratio of the real θ_m to the real θ_{mi}; the angle is equal to the phase angle of θ_m with respect to θ_{mi}. Thus it is possible to obtain the amplitude and phase of the frequency response from equation (26). Writing amplitudes as $|\theta_m|$ and $|\theta_{mi}|$ and letting ϕ be the phase angle, equation (26), yields:

$$\frac{|\theta_m|}{|\theta_{mi}|} = \frac{S_1}{\sqrt{(2Z_1T_1f)^2 + [1 - (T_1f)^2]^2}}, \tag{27}$$

$$\phi = -\tan^{-1}\frac{2Z_1T_1f}{1 - (T_1f)^2}. \tag{28}$$

The frequency response is seen to be completely defined by the three parameters S_1, Z_1, T_1.

S_1 may be termed the "slip" of the transmission. It represents the lost motion involved because of leakage and output shaft damping. T_1 is the natural period of the transmission. If a frequency equal to the reciprocal of this period is impressed and Z_1 is very small, a high resonant peak will occur in the output. This resonant effect is primarily caused by the elasticity of the fluid and the inertia. As a high resonant peak is a condition to be avoided in a servomotor, Z_1 must be made reasonably large. If Z_1, which is called the damping ratio, is equal to unity, no resonant peak will occur. Ordinarily, for satisfactory operation of hydraulic transmissions as servomotors, Z_1 should have a value between 0.5 and 1.0. However, applications do arise where it is profitable to go outside this range and such cases should have Z_1 adjusted to suit. As Z_1 is primarily a function of leakage and inertia, it may be adjusted by adjusting the leakage.

T_1 is what primarily determines the useful frequency range of the transmission. To see how T_1 affects the useful frequency range, consider the transmission in its application to servomechanisms. Almost invariably any servomechanism using a hydraulic servomotor is of the type that has no position error. This means that the maximum phase shift available for the servomotor and all the other components must be less than 90 degrees at the frequency where the overall amplitude ratio has fallen to unity if the system is to be stable. In view of this, it would certainly seem as if the servomotor would never have phase shifts as large as 45° in the useful frequency range. Defining any frequency for which the phase shift exceeds 45° lagging to be outside the useful range, the limiting frequency of useful transmission can be found from equation (28) if Z_1 is known. The limiting condition is (since tan 45° equals 1):

$$f_{45} = \frac{\sqrt{Z_1^2 + 1} - Z_1}{T_1}. \tag{29}$$

In the range $\frac{1}{2}$ to 1 for Z_1, $\sqrt{Z_1^2 + 1} - Z_1$ varies from 0.62 to 0.41.

Taking 0.5 as a rough average value, it is apparent that *the limiting frequency of transmission is approximately one-half of the natural frequency*, $1/T_1$. This, in practice, is found to be a high limit and difficult to obtain. Usually transmission frequencies are limited to much lower values so the "one-half natural frequency limit" might be called a "maximum upper limit."

The equations governing the behavior of the hydraulic transmission are non-linear. It was only by making approximations that a linear equation such as (22) could be obtained. It is therefore questionable if any of the results of equation (22) including the predicted frequency response given by equations (27) and (28) are valid. Only comparison of test results with predicted results can answer this. The remainder of the report will be given to calculations of the frequency response parameters for actual apparatus, comparisons of actual performance with predicted behavior, and a discussion of the discrepancies.

DISCREPANCIES BETWEEN PREDICTED AND MEASURED PERFORMANCE.

7. Transmission Calculations and Simplified Parametric Equations.

Test information will be presented on three hydraulic transmissions termed Transmission I, Transmission II and Transmission III. The parameters S_1, T_1 and Z_1 for Transmission I are accurately calculated using equations (7), (8), (9), (10), (11), (12), (13), (14), (23), (24), (25) and the information given in the appendix under "Transmission Data." The results are presented in Tables I and II. On the basis of this experience, it was found that only the items preceded by an asterisk in Tables I and II affected the resultant values of S_1, T_1 and Z_1 appreciably. All items not preceded by an asterisk can be assumed equal to zero and the resultant parameters will not differ from their accurately calculated values by more than two per cent.

Because of the simplification of expression that results when the equations for S_1, T_1 and Z_1 are written using the "important" quantities only, they will be repeated below omitting all quantities not marked by an asterisk in Tables I and II:

$$S_1 = 1.00, \tag{30}$$

$$T_1 = 2\pi\sqrt{B_2 J_0}. \tag{31}$$

$$Z_1 = \pi\frac{(B_1 + G_1)J_0}{T_1}. \tag{32}$$

Or in terms of the more fundamental constants:

$$S_1 = 1.00, \tag{33}$$

$$T_1 = \frac{2\pi\sqrt{C_h V_t J_0}}{D_m}, \tag{34}$$

TABLE I.

Fundamental Transmission Constants.

Symbol	Units	Values		
		Trans. I	Trans. II	Trans. III
L_{ep}	in.^{5} lb.$^{-1}$ sec.$^{-1}$	2.1×10^{-4}	—	—
L_{ip}	in.^{5} lb.$^{-1}$ sec.$^{-1}$	0.1×10^{-4}	—	—
L_p	in.^{5} lb.$^{-1}$ sec.$^{-1}$	2.2×10^{-4}	—	4.0×10^{-4}
L_c	in.^{5} lb.$^{-1}$ sec.$^{-1}$	0	0	0
L_{em}	in.^{5} lb.$^{-1}$ sec.$^{-1}$	2.1×10^{-4}	—	—
L_{im}	in.^{5} lb.$^{-1}$ sec.$^{-1}$	0.1×10^{-4}	—	—
L_m	in.^{5} lb.$^{-1}$ sec.$^{-1}$	2.2×10^{-4}	—	6.0×10^{-4}
$*L_t$	in.^{5} lb.$^{-1}$ sec.$^{-1}$	4.4×10^{-4}	3.67×10^{-4}	1.0×10^{-3}
G_c	in.$^{-5}$ lb.^{1} sec.^{1}	2.0×10^{-2}[c]	—	—
S_c	in.$^{-5}$ lb.^{1} sec.^{2}	1.0×10^{-2}	—	—
V_p	in.^{3} lb.^{0} sec.^{0}	0.62×10^{0}	—	1.71
V_c	in.^{3} lb.^{0} sec.^{0}	3.00×10^{0}	—	2.677×10^{1}[a]
V_m	in.^{3} lb.^{0} sec.^{0}	0.22×10^{0}	—	2.52
$*V_t$	in.^{3} lb.^{0} sec.^{0}	3.84×10^{0}	1.65×10^{0}	3.10×10^{1}
$*C_h$	in.^{2} lb.$^{-1}$ sec.^{0}	4.00×10^{-6}	4.00×10^{-6}	4.00×10^{-6}
$*K_p$[b]	in.^{6} lb.$^{-1}$ sec.^{0}	1.54×10^{-7}	2.45×10^{-7}	3.88×10^{-7}
K_m	in.^{5} lb.$^{-1}$ sec.^{0}	2.74×10^{-7}	—	—
F_m	in.^{1} lb.^{1} sec.^{1}	9.3×10^{-3}	—	—
$*J_m$	in.^{1} lb.^{1} sec.^{2}	0.07×10^{-2}	1.44×10^{-3}	1.47×10^{-2}
F_0	in.^{1} lb.^{1} sec.^{1}	9.3×10^{-3}	—	—
$*J_0$	in.^{1} lb.^{1} sec.^{2}	1.66×10^{-2}	6.62×10^{-2}	1.81×10^{0}
$*D_m$	in.^{3}	6.02×10^{-2}	9.54×10^{-2}	3.84×10^{-1}
$*\theta_p$	(rad.) sec.$^{-1}$	4.71×10^{2}	3.66×10^{2}	3.66×10^{2}

* Quantities not so marked are second order.

[a] Including volumes within special valve fittings.

[b] Value for zero stroke.

[c] Linear up to about 0.20 full speed; turbulent flow at higher speeds.

TABLE II.

Derived Transmission Constants.

Symbol	Units	Values		
		Trans. I	Trans. II	Trans. III
A_1	dimensionless	1.3×10^{-5}	—	—
A_1'	dimensionless	1.3×10^{-5}	—	—
A_2	sec.^{1}	7.0×10^{-6}	—	—
A_2'	sec.^{1}	7.0×10^{-6}	—	—
A_3	sec.^{2}	2.4×10^{-7}	—	—
$*B_1$	in.$^{-1}$ lb.$^{-1}$ sec.$^{-1}$	1.21×10^{-1}	4.03×10^{-2}	6.79×10^{-3}
$*B_2$	in.$^{-1}$ lb.$^{-1}$ sec.^{0}	4.24×10^{-3}	7.25×10^{-4}	8.42×10^{-4}
C_1	sec.^{1}	6.9×10^{-9}	—	—
C_2	sec.^{2}	3.4×10^{-9}	—	—
$*D_1$	in.$^{-1}$ lb.$^{-1}$ sec.^{0}	4.25×10^{-5}	2.70×10^{-5}	2.63×10^{-6}
E_1	in.$^{-1}$ lb.$^{-1}$ sec.^{0}	1.78×10^{-4}	—	—
$*G_1$[a]	in.$^{-1}$ lb.$^{-1}$ sec.$^{-1}$	0.14×10^{-1}	0.69×10^{-2}	0.68×10^{-4}
F_4	in.^{2} lb.$^{-1}$ sec.$^{-1}$	7.0×10^{-3}	—	—
S_1	dimensionless	0.999	—	—
$*T_1$	sec.	5.27×10^{-2}	4.35×10^{-2}	2.45×10^{-1}
$*Z_1$	dimensionless	0.136	0.226	0.173

* Quantities not so marked are second order.

[a] Based on 0.7 $D_1\theta_p$ as a proper value for low speed conditions.

$$Z_1 = \frac{\pi}{T_1}\left[\frac{L_t}{D_m^2} + \frac{k_1\theta_p C_h(V_{cl} + \frac{1}{2}V_{sw})\frac{N_p}{2\pi}}{D_m^2}\right]J_0, \tag{35}$$

$$Z_1 = \frac{\left[L_t + k_1\theta_p C_h(V_{cl} + \frac{1}{2}V_{sw})\frac{N_p}{2\pi}\right]\sqrt{J_0}}{2D_m\sqrt{C_h V_t}}. \tag{36}$$

$V_{cl} + \frac{1}{2}V_{sw}$ represents the volume of oil commutated each time a cylinder passes from the low pressure port to the high pressure port; k_1 is an arbitrary constant to approximate the effect of compressibility leakage (it was assumed to have the value 0.7 in making the calculations of Table II). This constant, k_1, multiplied by the compressibility leakage factor for the pump is supposed to obviate accounting for the compressibility leakage of the motor. In all calculations the compressibility leakage of the pump is calculated for zero or very small strokes.

Generally, sufficient accuracy and a considerable reduction of work results if equations (34) and (36) are used. Cases may arise, however, when the simpler equations do not yield sufficiently accurate results and these cases must be guarded against by making spot checks. It will be noted that S_1 equals unity primarily because F_0 was neglected (A_1' is usually extremely small). This assumption is valid for most cases of pure inertia loading as the viscous damping of the hydraulic motor is ordinarily quite small.

The parameters for Transmissions II and III are calculated on the basis of equations (31) and (32). The fundamental information is given in the appendix and the results are presented in Tables I and II.

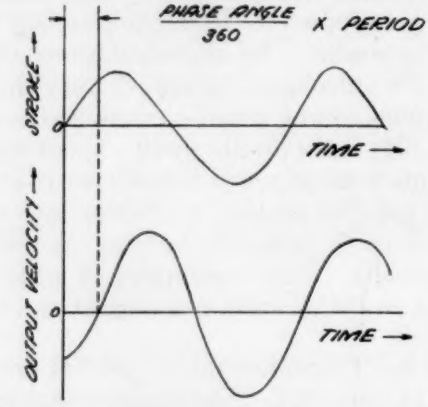
8. Experimental Measurements of Parameters.

In order to measure the parameters, frequency response curves were taken on the several transmissions. The transmissions were driven by electric motors at substantially constant speed and loaded with flywheels corresponding to the inertia property of their respective loads.

It is not to be inferred that the experimental data thus obtained is typical of the transmissions when used as servomotors driving their "application" loads. Only the inertia property of the application load is duplicated and even this not very accurately, since most loads met in practice have many inertias coupled to the servomotor through varying amounts of backlash, friction and elasticity. In addition, load torques caused by wind, recoil or other factors are not represented. The data obtained in the frequency response tests recorded here, rather than being indicative of performance under application conditions, should be regarded as the raw material from which the so-called "measured values" of the transmission parameters may be calculated. The meas-

ured transmission parameters then can be used in the overall equations of the servosystem in place of the calculated parameters for purposes of calculating its response. The measured parameters (if they can be obtained) have an advantage over frequency response data even if the loads are truly represented. This advantage is that they are principally characteristic of the transmission rather than of the transmission and load combined. Thus they are much more useful in calculating the effects of changes in the load or in any other parts of the system.

To take a frequency response curve of a transmission it is necessary to vary the pump stroke sinusoidally with time. The output velocity is measured and its amplitude and phase are compared with the stroke.



RELATIONSHIP OF STROKE AND OUTPUT VELOCITY AT A PARTICULAR FREQUENCY DURING FREQUENCY RESPONSE RUN

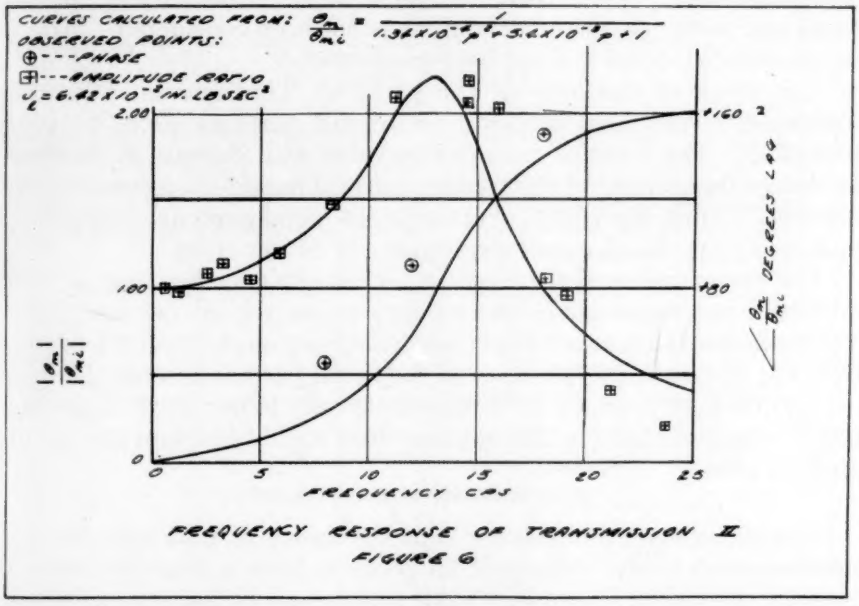
FIGURE 5

If the transmission were ideal, the output velocity would be instantaneously proportioned to the stroke providing the pump drive speed is constant. Therefore the amplitude ratio of the transmission response is the ratio of the amplitude of the output velocity to the amplitude of the stroke multiplied by a constant. The phase angle, ϕ, is the phase angle of the output velocity less the phase angle of the stroke (see Fig. 5).

The mechanism used to vary the stroke of the transmission is usually a hydraulic servo consisting of: a power cylinder with a piston linked to the pump so as to control its displacement; a valve assembly arranged to cause the piston to follow and amplify the motions of the valve; and a source of oil at substantially constant pressure to supply the valve and cylinder combination. By superimposing a vibratory motion upon the

signal supplied to the valve, dither is introduced. The input to the valve is varied sinusoidally by a scotch-yoke or similar device.

The pump stroke is measured by means of an electrical strain gauge, selsyn or potentiometer suitably coupled. The output velocity of the motor is measured by an electrical tachometer. The two resultant electrical signals are compared directly on a cathode ray oscillograph. Alternatively, they may be measured as functions of time on a bifilar oscillograph recording on photographic film or a direct-inking oscillograph recording on paper. In any event it is by comparison of the electrical signals that the amplitude ratio and phase angle are obtained. (It is sometimes possible to use direct mechanical means of comparison but this is usually inconvenient.)



FREQUENCY RESPONSE OF TRANSMISSION II
FIGURE 6

After making the necessary measurements, the data are generally plotted up as shown in Fig. 6. In order to obtain the measured values of the parameters, a second order differential equation is sought which gives amplitude and phase curves which best fit the plotted points. The coefficients of this equation are ordinarily determined by trial and error (if the data are extremely accurate more precise methods can be used). The precision of fit of the "best fit" differential equation is indicated in Fig. 6 by the scattering of the observed points away from the curves.

From the coefficients of the "best fit" differential equation the measured values of T_1 and Z_1 are computed. Table III presents for comparison the predicted and measured values of T_1 and Z_1 for the three transmissions.

TABLE III.

Predicted and Measured Parameters Compared.

Parameter	Trans. I	Trans. II	Trans. III
T_1 (predicted) sec.	5.27×10^{-2}	4.35×10^{-2}	2.45×10^{-1}
T_1 (measured) sec.	8.50×10^{-2}	7.27×10^{-2}	2.84×10^{-1}
Z_1 (predicted)	0.153	0.226	0.173
Z_1 (measured)	1.14	0.234	0.332

9. Discrepancies Between Predicted and Measured Natural Periods.

From Table III it will be noticed that the measured periods are generally greater than the predicted periods. This naturally raises the question of whether all the factors affecting the predicted period have been considered. The answer, of course, is no. Elasticity of the metal pump and motor parts and conduits has not been considered. Also the compressibility of oil has not been confirmed.

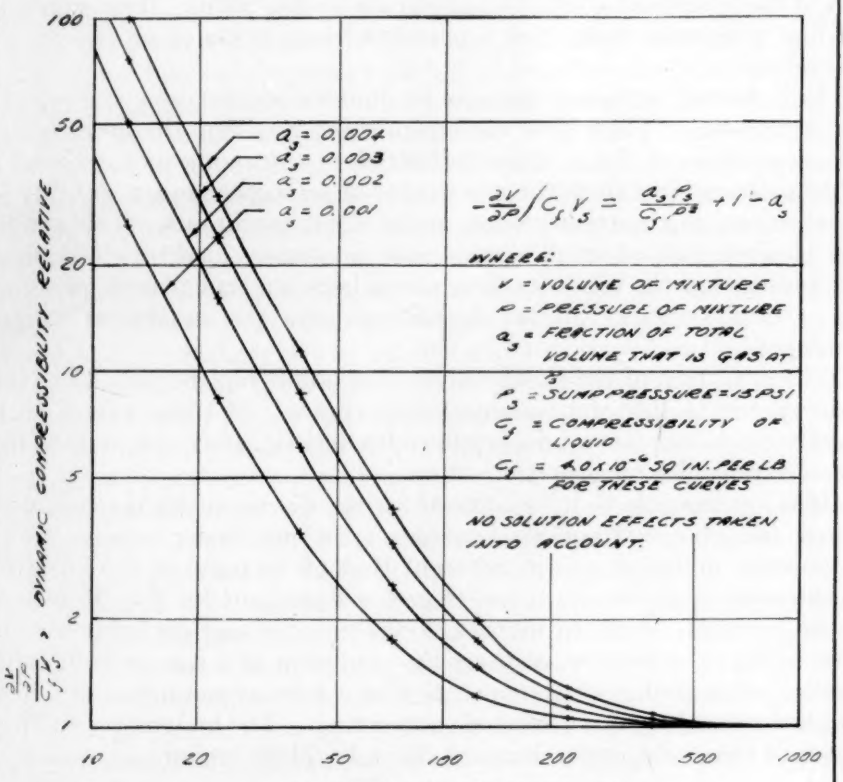
The effect of elasticity of the parts for Transmission I has been computed. The effect of pump and motor parts seems to be wholly negligible. The effect of conduit expansion with increase in pressure is to change the computed value of the natural period by approximately 8 per cent. Thus, the effects of elasticity of metal parts are not sufficient to account for the observed discrepancy of 61 per cent.

For Transmission II the elasticity of metal is even less than in Transmission I but the observed discrepancy is greater (67 per cent.). For Transmission III, a much larger and relatively more heavily loaded one than the first two, the elasticity of the metal parts is accountable to the extent of 1.1 per cent. for a discrepancy of only 16 per cent. Can the effective compressibility of the working fluid possibly explain this peculiar state of affairs?

10. Compressibility Variation with Pressure.

Evaluations of the average compressibility of hydraulic fluids by measurement of the volume of oil released from a chamber when the pressure is reduced from a high value to atmospheric pressure are possible even with crude equipment. However, evaluations of the average compressibility when the pressure difference is low are more difficult. Repeated measurements on the oils listed in the appendix under "Transmission Data—Fluids" give the value of the average compressibility for a pressure range of 0 to 2,000 pounds per square inch gauge as 4.0×10^{-6} square inches per pound. It was noticed, however, that oil samples taken from hydraulic transmissions had slightly larger compressibilities than samples of the same oils taken from the original containers. This seemed to confirm the suspicion that entrained gases were lowering the effective compressibilities of the oils. For example, measurements indicated that the fluid in Transmission I was composed of approximately 0.2 per cent. gas and 99.8 per cent. oil by volume at atmospheric pressure.

Figure 7 is a graph of the relative "dynamic compressibility" as a function of pressure and amount of gas at atmospheric pressure. Dynamic compressibility is defined as $(-1/V_s)(\partial V/\partial P)$ where V_s is the volume at sump pressure. Isothermal compression of the gas is assumed and it is supposed that there are no solution effects. It is appar-



DYNAMIC COMPRESSIBILITY AS A FUNCTION OF PRESSURE AND AMOUNT OF ENTRAINED GAS

FIGURE 7

ent from these curves that the compressibility increases markedly at low pressure if entrained gas is present.

The effect of gas is to make the elasticity of the transmission very non-linear at low pressures (torques). Dither helps to reduce these non-linear effects but cannot be expected to eliminate them if the amount of gas is very large.

From their construction Transmissions I and II could be expected to have more entrained gas than Transmission III. In the first two transmissions the oil sump was integral with the pump housings. As a result sump oil and air are continuously mixed by the high speed rotation of the cylinder barrel and piston rods of the pump. Transmission III had a separate sump which permitted separation of oil and air to take place.

The effect of entrained gas may be diminished by raising the replenishing pressure. The higher the replenishing pressure, the lower is the compressibility of the mixture admitted to the pump-motor circuit.[2] What matters most from the standpoint of servomechanism stability is the elasticity of the transmission under worst conditions. This condition is often that of small signals with no steady load on the motor. The pressure in the conduits then varies between replenishing pressure and a small amount above, depending upon the maximum torque developed.

The operation of the check valves that admit replenishing oil to the main circuit is also of paramount importance. If these valves stick slightly or do not have an abrupt characteristic, they can reduce the effective replenishing pressure.

It is not possible to fully account for the discrepancies between predicted and observed periods because of the non-linear nature of the phenomena involved. It is apparent that an increase in the effective compressibility of the oil of less than 4 will account for the differences in the periods. Such an increase looks possible and probable on the basis of Fig. 7. In some respects the operation of a transmission with "gassy" oil is similar to the operation of a torsion pendulum in which backlash exists between the inertia and spring. The low compressibility region of the transmission is much like a backlash region.

11 Discrepancies Between Predicted and Measured Damping Ratios.

The differences between predicted and measured damping ratios are large except in the case of Transmission II. Reference to Table III shows that the percentage errors are 650, 3, and 92 for Transmissions I, II, and III respectively. In the case of Transmission I, later measurements showed that its leakage varied greatly with output speed. This

[2] Dither considerations limit the possible increase in replenishing pressure. Equations 6 and 16 show that the $\pm F_4 (P_0 - P_s)$ term, which must be "covered up" by dither, increases with P_0, the replenishing pressure.

is an indication of a defective transmission; the parts were probably worn because of dirty oil or other abuse. Therefore no attempt will be made to account for the discrepancy of Transmission I.

The leakage of Transmission II was measured by taking static torque speed curves prior to running the frequency response tests, and was found to be relatively invariant with speed indicating a good transmission. This was to be expected since it had just been reconditioned by its manufacturer. At first glance it would be thought that the agreement of prediction and test with respect to the damping ratio was satisfactory. This is not the case however. Reference to equation (35) shows that the ratio of the effective leakage to the period is proportional to Z_1. If the period observed is greater than the predicted period, and the predicted and observed damping ratios, Z_1, have the same values, then the effective leakage observed must be greater than the predicted value. But the "predicted" value was measured by a static test. To explain this increase of some 60 per cent. in leakage it must be remembered that the effective leakage is composed of viscous leakage and compressibility leakage. The conditions of the prediction include a constant compressibility. The variation of compressibility is one factor explaining the difference in periods. It may also account for as much as a 45 or 50 per cent. increase in effective leakage because of its influence on the compressibility leakage. The variation in viscosity of the fluid between the leakage test and the frequency response test may well have been of the order of 15 or 20 per cent. since the viscosity of the fluid varies of the order of 1 per cent. per degree Fahrenheit and the temperatures during both tests were not well controlled. This would alter the viscous leakage by the same percentage. Therefore, considering compressibility leakage and viscous leakage together it seems possible to account for the damping ratios listed in Table III for Transmission II.

Transmission III has an observed damping ratio about 92 per cent. greater than the predicted ratio. No attempt will be made to resolve this difference since the leakage data upon which the predicted damping ratio is based are subject to doubt. The manufacturers catalogue data had to be depended upon and these are merely representative. Variations of particular used equipments from the average of new machines could easily account for the difference.

It will be noticed that predicted and observed damping ratios in all cases (except the observed ratio of Transmission I) are well below the values ordinarily desired for servomotors. Some means should be incorporated in future transmissions for controlling the damping ratio and compensating for temperature changes if they are to operate over anything but a narrow range of ambient temperatures. Temperature changes have definitely been proved to affect the performance of transmissions, both by tests of overall servomechanisms incorporating them as well as by tests of transmissions by themselves.

12. Conclusions.

The performance of a hydraulic transmission usually may be expressed (at least approximately) in terms of two parameters: the natural period and the damping ratio.

The natural period of a transmission is principally a function of the total inertia on the output shaft, the volume of oil subject to high pressure, the compressibility of the working fluid, and the displacement of the motor. It is desirable for the transmission to have as short a natural period as possible, consistent with other design considerations, in order that the useful frequency range of the servomechanism into which it is incorporated be as high as possible. Furthermore, the stability of the overall servomechanism may be unsatisfactory, or an excessive reduction of performance may be required to hold the stability within perscribed limits, if the natural period at any operating point is too great. The effective compressibility of the working fluid may rise considerably in the low pressure region if gases are entrained. A high replenishing pressure relative to sump pressure and proper sump design should aid in controlling the amount of gas entrained and the effect of such gas on performance.

The damping ratio of a transmission should ordinarily be maintained somewhere between 0.5 and 1.0 if the maximum performance of the overall servomechanism embodying it is to be obtained. It is generally found that the damping ratios of transmissions in current use are less than 0.5. Since the damping ratio is largely a function of the effective leakage, the output shaft inertia, and the natural period of the transmission, it may be controlled by varying the leakage. Generally the leakage needs to be increased. Therefore it should be desirable to incorporate in future transmissions some form of damping ratio control (with temperature compensation if necessary).

The parameters which determine transmission performance may be measured by observing the frequency response of the transmission. Pure inertia loading of the transmission is generally most satisfactory if the frequency response information is to be used for parameter determination. Operation of the transmission for any load may be computed, once the parameters for inertia loading have been obtained, on the basis of the transmission constants so determined.

APPENDIX.

13. Effect of Pump Speed Fluctuations.

In section 8 it was assumed in the discussion of frequency response that the pump speed remained substantially constant. If the pump is driven by an electric motor, its speed will generally not be constant but will vary with load. The load reflected back to the electric motor will depend upon the stroke, S_p. If the stroke is small, the usual condition

in making frequency response tests, very little load additional to standby losses is placed on the electric motor and the speed may be expected to remain nearly constant. What happens when the stroke is large can be determined by extending the analysis of sections 4, 5 and 6.

Assuming the hydraulic torque on the pump is the pressure difference multiplied by the displacement and using equations (1) and (4) for the pressure difference, an expression for the driving torque, M_{em}, required by the pump may be written:

$$M_{em} = 2G_cD_mD_{pm}S_p\dot{\theta}_m + 2S_cD_mD_{pm}S_p\ddot{\theta}_m + \frac{D_{pm}}{D_m}S_pM_m + J_p\ddot{\theta}_p + F_p\dot{\theta}_p. \quad (37)$$

Equation (20) gives the relationship between θ_p, S_p, M_m, θ_m if θ_{mi} is replaced by the original quantities:

$$\frac{D_{pm}}{D_m}S_p\theta_p = A_3\dddot{\theta}_m + A_2'\ddot{\theta}_m + (A_1' + 1)\dot{\theta}_m + B_2\dot{M}_m + (B_1 + G_1)M_m. \quad (38)$$

If M_m can be related to θ_m and M_{em} to θ_p, it is theoretically possible to solve a set of equations, composed of equations (37), (38) and these relationships, for any variable as a function of time. The resultant equation for θ_m is impossible to solve analytically because of product terms involving the derivatives of stroke and output shaft angle. However, in a general way, it can be seen that at small strokes the equation for θ_m tends toward the linear one given as equation (20).

The effect of large stroke angles on transmission performance can be seen by considering a degenerate case. Suppose the pump driveshaft's total inertia equals the total inertia on the motor shaft. Assume no friction and no leakage or elasticity. Further let the speed of the motor equal that of the pump at full stroke. If the power to the electric motor is turned off, the system has a certain amount of stored energy. For example, if the stroke is zero this energy is equal to the kinetic energy of the pump shaft inertia. If the stroke is increased to its full value the same energy will be divided equally between the pump and motor inertia. This means the pump speed would drop to $1/\sqrt{2}$ of its former value. If the stroke is varied sinusoidally, the pump speed will vary at twice the stroke frequency in a non-sinusoidal manner; its maximum deviation from the average of the extreme speeds will be 17.2 per cent. The motor speed also will vary in a non-sinusoidal manner, but its wave form will be like a sinusoid except more rounded, with crests 85.4 per cent. as high as the pure sinusoids that would exist if the pump speed were held constant at the mean of its extreme values.

Since the electric motor usually has an inertia of the same order of magnitude as the effective load inertia, and since energy will be added

to or subtracted from the system when the electric motor is connected to the power mains, the pump speed variations generally will be small enough to have negligible effect on the transmission's characteristics, in so far as they bear upon servomechanism operation, with the possible exception of stability at large strokes. At small strokes equation (20) or the equivalent is satisfactory for computing servomechanism stability. Without presenting the mathematics, it is difficult to predict the effect of large strokes on stability. Ordinarily, conditions are such that the system stability is equal to or greater than that existing at small strokes, but if analysis or test shows that the large stroke stability is for some reason diminished, it may be restored by the addition of damping windings to the electric motor, a flywheel to the pump shaft, or a similar expedient. An example of a system where such precautions might be necessary is one in which a synchronous motor is used to drive the pump.

14. Precautions in Making Frequency Response Tests.

Anyone who has made frequency response tests on hydraulic transmissions automatically exercises certain precautions. Since the transmission is fundamentally a non-linear device, the output will not be sinusoidal, for a sinusoidal input. The amplitude ratio and phase angle (to the extent that they can be defined in such a case) will vary with input amplitude as well as with frequency. The extraordinary finding is that the output wave form is so nearly sinusoidal. Nevertheless, it is essential that the input amplitude be carefully chosen and recorded with the rest of the data if repeatable results are desired. The actual value of the input amplitude to be used depends upon the purpose of the data.

Ordinarily, if the data are to be used for establishing transmission parameters for use in servomechanism stability studies, it is well to take two sets, one at as low amplitudes as possible (limited by measurement technique and "noise"), and another at as large amplitudes as possible (limited by maximum pressure). It is also desirable to make tests at large strokes as well as at small strokes. (This is done by measuring the variations of stroke and velocity about their mean values.)

The input amplitude will usually have to vary with frequency. At the lower frequencies, inputs that give substantially constant output velocity amplitudes are used (to avoid excessive strokes). At the higher frequencies, the input is adjusted to give approximately constant output acceleration amplitude (to avoid excessive pressures).

While making a frequency response test, much useful auxiliary information can be gathered on replenishing pressure, pump speed, leakage, fluid temperature and viscosity, etc., if suitable instruments are available and the test schedule is properly planned. Replenishing pressure and fluid temperature should be checked because they have important effects on the frequency response.

15. Transmission Data.

The following tables present the transmission data upon which all calculations of this paper are based.

TABLE IV.

Transmission Data—Pumps.

Item	Information	Units	Transmission I	Transmission II	Transmission III
1	Manufacturer		Vickers, Inc.	Vickers, Inc.	Vickers, Inc.
2	Model		AA 16850 A (double pump unit)	16800 complete transmission	PV-2003
3	Serial No.		141880 (azimuth)	Servo Lab. No. 1	Unit in housing No. T-163, A132
4	Approx. power	h.p.	4.5	2.2	8.5
	At Speed	r.p.m.	4500	3500	1750
	At Pressure	p.s.i.	1250	1200	2000
	At Stroke	degrees	25	10	30
5	Weight dry	lbs.	27, double pump unit with auxiliaries approx.	16.6, complete transmission measured	30, catalogue value
6	Maximum rated pressure [1]	p.s.i.	1000	1200	2000
7	Maximum rated speed [1]	r.p.m.	Unknown	Unknown	Unknown
8	Maximum displacement	cu. in. / rad.	0.0509	0.0332	0.153
	At stroke	degrees	25	10	30
9	Volume subject to high pressure	cu. in.	0.62	Only total oil vol. known	1.71
10	Minimum cylinder clearance	cu. in.	0.0117 (estimated)	0.0399 (estimated)	0.019 (estimated)
11	Max. displacement per piston per rev.	cu. in.	0.0457	0.0298	0.137
12	No. pistons		7	7	7
13	Dia. pistons	in.	0.350	0.406	0.469
14	Total leakage [2]	cu. in. / sec.	0.22 @ 1000 p.s.i.	Only overall leakage known	0.4 @ 1000 p.s.i.–0.8 @ 2000 p.s.i.
	Direct leakage	cu. in. / sec.	0.21 @ 1000 p.s.i.	Unknown	Unknown
	Differential leakage	cu. in. / sec.	0.01 @ 1000 p.s.i.	Unknown	Unknown
15	Rotational loss [3]	h.p.	0.700 for two pumps and replenisher	Unknown	Unknown
16	Inertia	in. lb. sec.^{2} / rad.	Unknown	Unknown	Unknown

[1] Maximum rated speed and pressure for continuous operation for normal life.

[2] Leakage at operating speed with fluid described in "Transmission Data—Fluid." All leakage data is approximate. Manufacturer's information used except for Transmission II where measurements on unit were made. All leakage data compensated for viscosity and temperature deviations. The breakdown of total leakage into direct and differential for Transmission I is estimated.

[3] At speed stated under approx. power, item 4.

TABLE V.

Transmission Data—Motors.

Item	Information		Units	Transmission I	Transmission II	Transmission III
1	Manufacturer			Vickers, Inc.	Vickers, Inc.	Vickers, Inc.
2	Model			AA 16850 B	See pump	MF-38-2
3	Serial No.			X 107919	Servo Lab. No. 1	Unit on Table 9T
4	Approx. power		h.p.	4.3	6.5	26
	At	Speed	r.p.m.	3600	3600	2200
		Pressure	p.s.i.	1250	1200	200
		Stroke	degrees	30	30	30
5	Weight dry		lbs.	2.5, measured	See pump	30, catalogue value
6	Maximum rated pressure [1]		p.s.i.	1000	Unknown	2000
7	Maximum rated speed [1]		r.p.m.	3600	Unknown	2200
8	Displacement		cu. in. / rad.	0.0602	0.0954	0.384
	At stroke		degrees	30	30	30
9	Volume subject to high pressure		cu. in.	0.22	Only total oil vol. measured	2.52
10	Cylinder clearance		cu. in.	0.0076 (estimated)	0.0120 (estimated)	0.048 (estimated)
11	Displacement per piston per rev.		cu. in.	0.0540	0.0855	0.344
12	No. pistons			7	7	7
13	Dia. pistons		in.	0.350	0.406	0.657
14	Total leakage [2]		cu. in. / sec.	0.22 @ 1000 p.s.i.	Only overall leakage known	0.6 @ 1000 p.s.i.-1.2 @ 2000 p.s.i.
	Direct leakage		cu. in. / sec.	0.21 @ 1000 p.s.i.	Unknown	Unknown
	Differential leakage		cu. in. / sec.	0.01 @ 1000 p.s.i.	Unknown	Unknown
15	Rotational loss		h.p.	0.2 approx. @ 3600 r.p.m.	Unknown	Unknown
16	Inertia		in. lb. sec.^{2} / rad.	0.000669, measured	0.00144 (estimated)	0.0147 (estimated)

[1] See corresponding note under "Transmission Data—Pumps."

[2] See corresponding note under "Transmission Data—Pumps."

TABLE VI.

Transmission Data—Fluid.

Item	Information	Units	Transmission I	Transmission II	Transmission III
1	Manufacturer		Standard Oil Co., N. J.	Standard Oil Co., N. J.	Standard Oil Co., N. J.
2	Trade designation		Univis J-43	Univis—40	Univis—54
3	Viscosity	centi-stokes	7.5 @ 158° F.	10 @ 150° F.	23 @ 140° F.
4	Specific gravity		0.84	0.84 approx.	0.84 approx.
5	Compressibility	in.^{2} / lb.	4.0×10^{-6} @ 0–2000 p.s.i.	4.0×10^{-6} @ 0–2000 p.s.i.	4.0×10^{-6} @ 0–2000 p.s.i.

TABLE VII.

Transmission Data—General.

Item	Information	Units	Transmission I	Transmission II	Transmission III		
1	Pump operating speed	r.p.m.	4500	3500	3500		
2	Ambient temperature	deg. F.	77	75 approx.	68 approx.		
3	Fluid temperature	deg. F.	158	150 approx.	140 approx.		
4	Conduit material		Seamless annealed copper tubing	No data on conduit; only total oil vol. measured	Seamless steel tubing [2]		
					Section I	Section II	Section III
5	Conduit length each [1]	in.	20	No data on conduit; only total oil vol. measured	68	12	12.6
6	Conduit inside dia.	in.	0.438	No data on conduit; only total oil vol. measured	0.440	0.560	0.370
7	Conduit outside dia.	in.	0.500	No data on conduit; only total oil vol. measured	0.630	0.750	0.500
8	Conduit volume each [1]	cu. in.	3.00	No data on conduit; only total oil vol. measured	10.33	2.95	1.36
9	Volume in fittings, each side [1]	cu. in.	Negligible	Included in total volume	12.12		
10	Replenishing pressure	p.s.i.	50 gauge	35 gauge	50 gauge approx.		
11	Load inertia	in. lb. sec.^{2}	0.0159	0.0648	1.80		
12	Load damping	in. lb. sec.	Negligible	Negligible	Negligible		
13	Total output shaft inertia	in. lb. sec.^{2}	0.0166	0.0662	1.81		
14	Total damping	in. lb. sec.	0.0093	Unknown	Unknown		
15	Total volume of high pressure oil [1]	cu. in.	3.84	1.65	31.0		
16	Overall leakage constant	in.^{5} lb.$^{-1}$ sec.$^{-1}$	4.4×10^{-4}	3.67×10^{-4}	1.0×10^{-3}		
17	Approx. max. power	h.p.	4.3	2.2	8.5		

[1] Slight differences in volumes (or lengths) of conduits, fittings, etc., communicating with the two pump connections have been averaged so the datum represents the mean volume (or length) of a high pressure oil path.

[2] Sections I and II connect pump and motor; section III connects main conduit to valve block containing relief valves.

16. Table of Symbols.

Below are summarized the symbols used in this paper with their units and definitions, or references to definitions. Under units, the exponents of $L^aF^bT^c$ are written in that order, where L is length, F is force and T is time.

Symbol	Units a	b	c	Definition or Reference
A_1	0	0	0	Equation (7)
A_1'	0	0	0	Equation (19)
A_2	0	0	1	Equation (8)
A_2'	0	0	1	Equation (19)
A_3	0	0	2	Equation (9)
a_s	0	0	0	Fraction of V_s that is gas
B_1	−1	−1	−1	Equation (10)
B_2	−1	−1	0	Equation (11)
C_1	0	0	1	Equation (12)
C_2	0	0	2	Equation (13)
C_h	2	−1	0	Effective compressibility of working fluid
C_f	2	−1	0	Compressibility of pure fluid
D_1	−1	−1	0	Equation (14)
D_m	3	0	0	Displacement per radian of motor
D_p	3	0	0	Displacement per radian of pump
D_{pm}	3	0	0	Maximum displacement per radian of pump
E_1	−1	−1	0	Equation (17)
F_2	1	1	1	Figure 4
F_3	1	1	1	Figure 4
F_4	2	−1	−1	Equation (15)
F_l	1	1	1	Damping constant of load
F_m	1	1	1	Damping constant of motor
F_0	1	1	1	Total damping constant of output shaft
F_p	1	1	1	Damping constant of pump
f	0	0	−1	Frequency
G_1	−1	−1	−1	Equation (19)
G_c	−5	1	1	Conduit resistance to flow
J_3	1	1	2	Figure 4
J_l	1	1	2	Inertia of load
J_m	1	1	2	Inertia of motor
J_0	1	1	2	Total inertia on output shaft
J_p	1	1	2	Inertia of pump
j	0	0	0	$\sqrt{-1}$
K_3	1	1	0	Figure 4
K_m	5	−1	0	Commutation loss coefficient of motor.
K_p	5	−1	0	Commutation loss coefficient of pump.
k_1	0	0	0	Arbitrary constant introduced in equations (35) and (36) to approximate effect of compressibility leakage
L_c	5	−1	−1	Differential leakage constant of conduits
L_{em}	5	−1	−1	Direct leakage constant of motor
L_{im}	5	−1	−1	Differential leakage constant of motor
L_{ep}	5	−1	−1	Direct leakage constant of pump
L_{ip}	5	−1	−1	Differential leakage constant of pump
L_m	5	−1	−1	Total leakage constant of motor
L_p	5	−1	−1	Total leakage constant of pump
L_t	5	−1	−1	Total leakage constant of transmission
M_m	1	1	0	Hydraulic moment (torque) of motor
M_{em}	1	1	0	Pump driving moment (torque)
N_p	0	0	0	Number of pistons
P_0	−2	1	0	Replenishing pressure

Symbol	Units a	b	c	Definition or Reference
P_1	−2	1	0	Pressure at conduit—pump connection (see Fig. 3)
P_2	−2	1	0	Pressure at conduit—motor connection (see Fig. 3)
P_3	−2	1	0	Pressure at conduit—motor connection (see Fig. 3)
P_4	−2	1	0	Pressure at conduit—pump connection (see Fig. 3)
P_s	−2	1	0	Sump pressure
p	0	0	−1	Operator d/dt
S_l	0	0	0	Transmission slip (see equation (23))
S_c	−5	1	2	Conduit resistance to rate of change of flow
S_p	0	0	0	Per unit stroke of pump
T_1	0	0	1	Transmission natural period (see equation (24))
t	0	0	1	Time
V_c	3	0	0	Volume of a conduit
V_{cl}	3	0	0	Volume of fluid in cylinder clearance
V_m	3	0	0	Volume of fluid subject to high pressure in motor
V_p	3	0	0	Volume of fluid subject to high pressure in pump
V_{sw}	3	0	0	Volume of fluid swept out of cylinder by piston
V_s	3	0	0	Volume of mixture of oil and gas at sump pressure
V_t	3	0	0	Total volume of fluid subject to high pressure
Z_1	0	0	0	Transmission damping ratio
θ_m	0	0	0	Motor shaft angle (radians)
θ_{mi}	0	0	0	Output shaft angle of ideal transmission
θ_p	0	0	0	Pump shaft angle
ϕ	0	0	0	Phase angle of transmission response
π	0	0	0	Ratio of circumference to diameter of circle

17. Bibliography.

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Electric Launching Device for Heavy Aircraft. (*Machinery*, Vol. 53, No 5.)—An electric launching devise is the latest answer to the problem of launching heavy bombers and jet-propelled aircraft from the decks of the Navy's carriers or from small landing fields whose limited runways restrict the movement of large aircraft. It is also being adapted to launch commercial airliners.

Developed by engineers of the Westinghouse Electric Corporation, the electric catapult—"Electropult"—is a huge induction motor, the stator of which has been rolled out flat and the rotor of which has been built into a shuttle car. When current is fed to the coils, the shuttle car speeds down the track and thus launches the aircraft. Running free the shuttle car can build up to a speed of approximately 220 miles per hour in slightly less than 500 feet; at this speed, it develops 10,000 hp. Calibration tests indicate that it can launch the largest planes operated by the Navy, with the exception of planes in the Mars class. It has launched a jet-propelled plane at 116 miles an hour in four and one-tenth seconds after a run of only 340 feet.

Two Electropults have been built for the Navy, the first installed at Mustin Field, Philadelphia, and the other at the Naval Air Test Center, Patuxent River, Maryland. The track of the latter unit is 1382 feet long. A Pratt & Whitney 1100-hp. aircraft engine drives a direct-current generator, which furnishes the power to operate the unit. A direct-current motor, connected to this generator, drives a 24-ton flywheel, which in turn, drives another generator. It is the energy stored in the flywheel that permits the generator to supply 12,000 kw. of electricity to the Electropult in the first few seconds of operation.

The chief advantage of this electric launching device over the conventional catapult is its gradual acceleration to launching speed. For commercial airliners, maximum acceleration would be about one "G" which means that a passenger would be pressed back in his seat by a force about equal to his own weight.

R. H. OPPERMANN.

ON THE TWO-DIMENSIONAL STEADY FLOW OF A COMPRESSIBLE VISCOUS FLUID FAR BEHIND A SOLID SYMMETRICAL BODY.*

BY

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INTRODUCTION.

Goldstein † solved the equations of steady flow of an incompressible viscous fluid in a wake far behind a solid symmetrical body by application of the method of successive approximations. Very far downstream in a wake, if the motion were steady, the assumptions and approximations of the laminar boundary layer theory would be valid. In the present paper this method was applied to the solution of the equations of steady flow of a compressible viscous fluid in a wake far behind a solid symmetrical body. The following equations are taken into account: equations of motion, continuity, energy and state. The coefficients of viscosity and thermal conductivity are assumed to be functions of temperature. Similarly as in Goldstein's work only the first two approximations are successful, whereas the third one is not successful.

I. 1. General equations of motion of a compressible viscous fluid and the basic laws.

The general equations of motion of a compressible viscous fluid in a three-dimensional flow are:

$$\begin{aligned}
\rho \frac{Du}{Dt} &= \rho X + \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}, \\
\rho \frac{Dv}{Dt} &= \rho Y + \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z}, \\
\rho \frac{Dw}{Dt} &= \rho Z + \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z},
\end{aligned} \tag{1}$$

* Presented at the Meeting of the American Mathematical Society, Applied Mathematics Session, Chicago, April 25–26, 1947.

† S. Goldstein, "On the Two-dimensional Steady Flow of a Viscous Fluid Behind a Solid Body—I." *Proceedings of the Royal Society of London*, Series A, Vol. 142, 1933, pp. 545–562. Also: "Modern Developments in Fluid Dynamics." Edited by S. Goldstein, Oxford, Clarendon Press, 1938.

$$p_{xx} = -p - \tfrac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial u}{\partial x},$$

$$p_{yy} = -p - \tfrac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial v}{\partial y}, \tag{1a}$$

$$p_{zz} = -p - \tfrac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z},$$

$$p_{yz} = p_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),$$

$$p_{zx} = p_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{1b}$$

$$p_{xy} = p_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

where the symbols used denote:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}, \tag{1c}$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}, \text{ etc.,} \tag{1d}$$

$$\vec{F} = -\operatorname{grad}\Omega + \vec{G}, \tag{1e}$$

$$-\operatorname{grad}\Omega = \vec{i}X + \vec{j}Y + \vec{k}Z, \tag{1f}$$

$$\vec{G} = \vec{k}g\left(\frac{\rho - \bar{\rho}}{\rho}\right),^{*} \tag{1g}$$

$$\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w - \text{velocity vector},$$

X, Y, Z—components of the extraneous forces $\vec{F}$ per unit mass at the point (x, y, z) at the time t,

p—pressure at the point (x, y, z),
ρ—mass density at the point (x, y, z),
μ—coefficient of viscosity.

Eq. (1) was derived by use of the principle of the conservation of momentum. Besides (1) the following equations have to be taken into account:

Continuity equation (law of conservation of mass):

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \tag{2}$$

* The assumption is the Z-axis coincides with the vertical axis.

The equation of the conservation of energy:

$$\rho \mathfrak{J} c_v \frac{DT}{Dt} + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \mathfrak{J}\nabla T + \Phi, \tag{3}$$

where the symbols used denote:

$\mathfrak{J}$—mechanical equivalent of heat,
c_v—specific heat at constant volume,
T—absolute temperature,
Φ—dissipation function,

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}, \tag{3a}$$

$$\nabla T = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right), \tag{3b}$$

k = coefficient of thermal conductivity,

$$\Phi = \tfrac{1}{2}\mu(e_{xx}^2 + e_{yy}^2 + e_{zz}^2 + 2e_{yz}^2 + 2e_{zx}^2 + 2e_{xy}^2) - \tfrac{1}{6}\mu(e_{xx} + e_{yy} + e_{zz})^2 = \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2\right]^*, \tag{4}$$

The equation of state:

$$p = \bar{R}\rho T. \tag{5}$$

For a perfect gas the specific heat at constant volume, c_v, and the specific heat at constant pressure, c_p, are constant. It is presumed that these coefficients are also constant in the present case although for real gases they depend upon the temperature and pressure.

I. 2. Two-dimensional steady flow with no extraneous forces.

Elimination of ρ from the equations of motion, continuity and energy by the use of eq. (5) gives in this case the following four equations:

Two equations of motion:

$$p\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \bar{R}T\left\{\frac{\partial p}{\partial x} - \frac{2}{3}\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] - \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]\right\} = 0, \tag{6}$$

* Goldstein, *loc. cit.*, Vol. II, p. 603.

$$p\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+\bar{R}T\left\{\frac{\partial p}{\partial y}-\frac{2}{3}\frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\right]-\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]\right\}=0, \quad (7)$$

Equation of continuity:

$$T\left[\frac{\partial}{\partial x}(pu)+\frac{\partial}{\partial y}(pv)\right]-p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=0, \quad (8)$$

Equation of energy:

$$\alpha p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)+pT\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\mathfrak{Z}T\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right]-\mu T\left\{\frac{4}{3}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right]+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\}=0, \quad (9)$$

with the notations:

$$\frac{\mathfrak{Z}c_v}{\bar{R}}=\frac{1}{(\gamma-1)}=\alpha, \qquad \gamma=c_p/c_v, \qquad \bar{R}=\mathfrak{Z}(c_p-c_v). \quad (9a)$$

I. 3. Coefficients of viscosity and heat conductivity.

In the most general case of flow of a compressible viscous fluid the coefficients μ, k, c_p and c_v are functions of pressure and temperature. Assuming, however, the invariability of the last two, one cannot neglect the variability of the first two coefficients. Since the coefficient k may be considered to be proportional to a product of μ and c_v, it is sufficient to estimate the variation of μ only. Because of the fact that the dependence of this coefficient on the pressure is very small for any change likely to occur in ordinary circumstances * one may take into account only the dependence of the temperature. For perfect gases μ varies as $\mu_0(T/T_0)^{\frac{1}{2}}$,† for real gases as $T^{0.7}$ or perhaps even more rapidly. For air, according to Grindley and Gibson, the viscosity may be expressed by the formula $\mu=\mu_0(1+at+bt^2)$, while Helmholtz, using Poiseuille's experiments, found that for water μ may be expressed as a power series in t with positive and negative coefficients.‡ Maxwell found

* Goldstein, *loc. cit.*, Vol. I, p. 7.

† A. Busemann, "Gasströmung mit laminarer Grenzschicht entlang einer Platte," *ZAMM*, Vol. 15, 1935, pp. 23–25.

‡ Russel A. Dodge, Milton J. Thompson, "Fluid Mechanics." McGraw-Hill Book Co., New York and London, 1937, p. 177. Also: H. Lamb, "Hydrodynamics." Cambridge, University Press, 1906, p. 536.

as the result of his experiments that the viscosity of air has the value $\mu = \mu_0(1 + at)$.* In the considerations below this coefficient will be assumed to be a polynomial or a power series in T with properly selected coefficients. The coefficient k will be assumed to be also a function of T in the form of a polynomial or a power series. This assumption is justified by the fact that k may be considered to be proportional to a product of μ and c_v or μ and c_p.†

I. 4. Assumptions.

The solution of the problem will consist of four stages: the initial and the first, second, and third approximations. Following the works of Goldstein and others, in the initial stage of calculations very far downstream in a wake, if the two-dimensional motion were steady, the assumptions and approximations of the laminar boundary layer theory would be valid. The pressure would be nearly constant across a section of the wake, the transverse velocity small in comparison with the longitudinal velocity, and the rate of change of the longitudinal velocity along the axis of the wake small compared with the rate of change across a section. The pressure gradient along the axis of the wake is also negligibly small. Consequently as a practical result the second equation of motion drops out. In the present paper in this stage of calculations the assumption is made that the temperature is constant across a section of the wake and its gradient along the axis is also negligibly small. The assumptions as concerning the two components of velocity remain the same as explained above. In the first approximation the energy equation will be used to calculate the temperature distribution and the equation of continuity will serve to give the vertical velocity component distribution. These values will be applied to obtain the second approximation of the longitudinal velocity component, etc. The second equation of motion will serve as a measure of the rate of change of the pressure. It will be shown that beginning from the second approximation this change may be expressed in a definite form. The first approximation does not give any change in pressure.‡

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* Lamb, *loc. cit.*, p. 537.

† Goldstein, *loc. cit.*, Vol. I, p. 10. In the cited formulae the temperature t is measured in degrees centigrade. In the book: E. Sänger, "Raketen-Flugtechnik," München und Berlin, 1933, Verlag von R. Oldenbourg, the following value is given for the air: $10^6\mu = 1.712 \times (1 + 0.003\,665t)^{\frac{1}{2}}(1 + 0.0008t)^2$. See also: H. L. Dryden, F. D. Murnaghan, H. Bateman, "Report of the Committee on Hydrodynamics," Bulletin of the National Research Council, No. 84. Washington, February 1932.

‡ For more details of the method of successive approximations see Goldstein's paper, *loc. cit.*

II. First approximation.

The symbols $\bar{u}$, $\bar{v}$, $\bar{\rho}$, $\bar{p}$, $\bar{T}$, $\bar{\mu}$, $\bar{k}$ will refer to the conditions in the undisturbed stream. As the initial values let:

$$T = \sim \bar{T}, \qquad u = \sim \bar{u}, \qquad v = 0, \qquad \mu = \sim \bar{\mu}. \tag{10}$$

As the first approximation let:

$$u = u_1 = \bar{u}(1 - w_1). \tag{10a}$$

With the above assumptions, and with the initial values, the first equation of motion takes the form:

$$\bar{p}\bar{u}\frac{\partial w_1}{\partial x} - \bar{R}\bar{T}\bar{\mu}\frac{\partial^2 w_1}{\partial y^2} = 0. \tag{10b}$$

The boundary conditions are that w_1 must tend to zero as x tends to infinity and must tend to zero exponentially as y tends to infinity. From symmetry w_1 must be an even function of y, so that $\partial w_1/\partial y$ must be zero when y is zero. Apply the following transformations and substitutions:

$$\bar{\nu} = \bar{\mu}/\bar{\rho}, \qquad \eta = y\left(\frac{\bar{u}}{2\bar{\nu}x}\right)^{\frac{1}{2}}, \qquad w_1 = A_1 x^{-\frac{1}{2}} e^{-c\eta^2}, \tag{11}$$

$$\frac{\partial w_1}{\partial x} = \frac{1}{2}\frac{1}{c}\frac{\bar{\nu}}{\bar{u}}\frac{\partial^2 w_1}{\partial y^2}.$$

w_1 is of the order $x^{-\frac{1}{2}}$ and in the first approximation only the terms of order $x^{-\frac{1}{2}}$ are kept. Solution of (10b) gives the value $c = \frac{1}{2}$. In the same way the equation of energy will be reduced. With the same initial values as bove, with $k = \sim \bar{k}$, $T_1 = \bar{T}(1 - z_1)$, as the first approximation, and keeping only the terms of order $x^{-\frac{1}{2}}$, one obtains:

$$c_1\frac{\partial^2 z_1}{\partial y^2} - c_2\frac{\partial z_1}{\partial x} = c_3\frac{\partial w_1}{\partial x} = -\tfrac{1}{2}c_3 A_1 x^{-\frac{1}{2}} e^{-\frac{1}{2}\eta^2}(1 - \eta^2), \tag{12}$$

$$c_1 = 3\bar{k}\bar{T}, \qquad c_2 = \alpha\bar{u}\bar{p}, \qquad c_3 = \bar{u}\bar{p}.$$

Preserve the same boundary conditions for z_1 as for w_1. Put:

$$z_1 = x^{-\frac{1}{2}} f_1(\eta). \tag{13}$$

Obtain:

$$c_1\bar{u}f_1'' + c_2\bar{\nu}(\eta f_1' + f_1) = -c_3 A_1\bar{\nu}e^{-\frac{1}{2}\eta^2}(1 - \eta^2). \tag{13a}$$

Put:

$$f_1 = e^{-\frac{1}{2}\eta^2} g_1. \tag{14}$$

Obtain:

$$(c_2\bar{\nu} - c_1\bar{u})(1 - \eta^2)g_1 + (c_2\bar{\nu} - 2c_1\bar{u})g_1' + c_1\bar{u}g_1'' = -c_3 A_1\bar{\nu}(1 - \eta^2). \tag{14a}$$

A solution of (14a) is:

$$g_1 = CA_1, \qquad C = c_3\bar{\nu}(c_1\bar{u} - c_2\bar{v})^{-1} = \bar{p}\bar{\nu}[\mathfrak{z}\bar{T}(\bar{k} - \bar{\mu}C_v)]^{-1}. \tag{15}$$

The coefficient C is a dimensionless one. Hence one has:

$$z_1 = A_1Cx^{-\frac{1}{2}}e^{-\frac{1}{2}\eta^2} = Cw_1. \tag{15a}$$

Putting into the equation of continuity (8) the values u_1, $v = v_1$, T_1, and keeping the terms of order $x^{-\frac{1}{2}}$ and omitting those of higher order, reduces it to the form:

$$\bar{u}(1 - C)\frac{\partial w_1}{\partial x} - Cv_1\frac{\partial w_1}{\partial y} - \frac{\partial v_1}{\partial y} = 0. \tag{16}$$

Hence v_1 is of the order x^{-1} and the omitting of the second term of order x^{-2} transforms (16) into:

$$\bar{u}(1 - C)\frac{\partial w_1}{\partial x} - \frac{\partial v_1}{\partial y} = 0. \tag{16a}$$

The boundary conditions are that v_1 must be zero when η or y is zero and when x and y tend to infinity. From symmetry v_1 is an odd function of y. The solution is:

$$v_1 = (C - 1)A_1\left(\frac{\bar{u}\bar{\nu}}{2}\right)^{\frac{1}{2}}x^{-1}\eta e^{-\frac{1}{2}\eta^2}. \tag{17}$$

The first approximate value of A_1 may be found if the value of the drag of the obtacle is known.*

The considerations of momentum show that the drag D_u per unit breadth of the obstacle is given by:

$$D_u = \int_{-\infty}^{+\infty}(\bar{\rho}\bar{u}^2 - \rho u^2)dy. \tag{18}$$

But in the first approximation $\bar{\rho}\bar{u} = \rho u$; hence:

$$D_u = \bar{\rho}\bar{u}^2\int_{-\infty}^{+\infty}w_1dy = 2\bar{\rho}\bar{u}^{\frac{1}{2}}A_1(\pi\bar{\nu})^{\frac{1}{2}}. \tag{18a}$$

A more precise value of A_1 may be obtained by introducing a certain refinement. Namely:

$$\rho u^2 = \bar{p}(\bar{R}\bar{T})^{-1}\bar{u}^2(1 - w_1)^2(1 - Cw_1)^{-1}$$
$$= \bar{\rho}\bar{u}^2(1 - 2w_1 + Cw_1 - 2Cw_1^2 + C^2w_1^2 - \cdots). \tag{19}$$

* The dimension of A_1 is (length)$^{\frac{1}{2}}$.

Keeping only the first three terms of the expansion gives the formula:

$$D_u = (2 - C)\bar{\rho}\bar{u}^2 \int_{-\infty}^{+\infty} w_1 dy.* \tag{20}$$

As the first approximations of μ and k one gets:

$$\begin{aligned} \mu_1 &= \mu_0(1 + aT_1 + bT_1^2 + \cdots) = \bar{\mu} - \mu_0(b_1 z_1 - c_1 z_1^2 + \cdots), \\ \bar{\mu} &= \mu_0(1 + a\bar{T} + b\bar{T}^2 + \cdots) = \mu_0 a_1, \\ b_1 &= (a\bar{T} + 2b\bar{T}^2 + \cdots), \text{ etc.}, \end{aligned} \tag{21}$$

$$\begin{aligned} k_1 &= \text{const. } c_p\mu_1 = \bar{k} - \text{const. } c_p\mu_0(b_1 z_1 - c_1 z_1^2 + \cdots), \\ \bar{k} &= \text{const. } a_1 c_p \mu_0 = \text{const. } c_p\bar{\mu}. \end{aligned} \tag{22}$$

In all the calculations below the constant will be included into the value c_p. Substituting into the second equation of motion the values u_1, v_1, T_1, μ_1, and neglecting all the terms of order higher than $x^{-\frac{1}{2}}$ gives the following result:

$$\bar{R}\bar{T}(1 - z_1)\frac{\partial p}{\partial y} = 0. \tag{23}$$

This result may be considered to confirm the initial assumptions (p = const. in each connection).

III. Second approximation.

In the second approximation put into the first equation of motion the values $u_2 = \bar{u}(1 - w_1 - w_2)$, v_1, T_1, and μ_1, and calculate the value w_2. Next put into the equation of energy the values u_2, v_1, $T_2 = \bar{T}(1 - z_1 - z_2)$, μ_1, and k_1, and calculate the value z_2. After that use the values u_2, $v = v_1 + v_2$, and T_2, in order to calculate v_2 from the

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* Emmons and Brainerd (see: H. W. Emmons, J. G. Brainerd, "Temperature Effects in a Laminar Compressible-Fluid Boundary Layer Along a Flat Plate." *Journal of Applied Mechanics*, Vol. 8, No. 3, Sept. 1941. Also: J. G. Brainerd, H. W. Emmons, "Effect of Variable Viscosity on Boundary Layers with a Discussion of Drag Measurements." *Journal of Applied Mechanics*, Vol. 9, No. 1, March 1942) calculated the drag of a plate at zero incidence in a compressible fluid and derived the formula:

$$D_u = c_D L\bar{\rho}\bar{u}^2, \qquad c_D = \frac{4U_0'}{(Re)^{\frac{1}{2}}}, \qquad U = \left(\frac{u}{\bar{u}}\right), \qquad Re = \left(\frac{\bar{\rho}\bar{u}L}{\bar{\mu}}\right), \tag{20a}$$

where L denotes the length of the plate and the subscript zero refers to the surface of the plate. The values of U_0' are given in the previously mentioned papers, *loc. cit.*, as functions of Prandtl's number P_r and compressibility number $(\gamma - 1)\bar{M}^2$. Hence from comparison of (18a) and (20a) one obtains:

$$A_1 = 2\left(\frac{L}{\pi}\right)^{\frac{1}{2}} U_0'. \tag{20b}$$

From comparison of (20a) and (20) one obtains:

$$A_2 = 2(2 - C)^{-1}\left(\frac{L}{\pi}\right)^{\frac{1}{2}} U_0'. \tag{20c}$$

equation of continuity. The substitution of all the values from the second approximation into the second equation of motion will give the second approximation of p. w_2 and z_2 will be of the order x^{-1}, v_2 of order $x^{-\frac{1}{2}}$. To calculate w_2, z_2, and v_2, one has to keep in all the equations used only the terms of the order x^{-2}, omitting all other terms. After all the necessary transformations the first equation of motion will give the following result:

$$\bar{\mu}\frac{\partial^2 w_2}{\partial y^2} - \bar{\rho}\bar{u}\frac{\partial w_2}{\partial x} = \tfrac{1}{2}A_1^2 x^{-2}(D_2\eta^2 - D_1)e^{-\eta^2}. \tag{24}$$

Dimensions of coefficients D_i are lb. sec./inch^3.

$$D_1 = [C\mu_0\bar{\mu}^{-1}(a_1 + b_1) - 1]\bar{u}\bar{\rho}, \tag{25a}$$

$$D_2 = C[\mu_0\bar{\mu}^{-1}(a_1 + 2b_1) - 1]\bar{u}\bar{\rho}. \tag{25b}$$

The boundary conditions are again that w_2 must tend to zero as x tends to infinity, and must tend to zero exponentially as y tends to infinity. Also $\partial w_2/\partial y$ must be zero when y is zero. z_2 must fulfill boundary conditions similar to w_2. Both w_2 and z_2 are even functions of η. v_2 must be zero when y is zero and when both x and y tend to infinity. It is an odd function of y.

Following Goldstein put:

$$w_2 = x^{-1}f_2(\eta), \tag{26}$$

and obtain:

$$f_2'' + \eta f_2' + 2f_2 = A_1^2(\bar{u}\bar{\rho})^{-1}(D_2\eta^2 - D_1)e^{-\eta^2}. \tag{26a}$$

Put:

$$f_2 = e^{-\frac{1}{2}\eta^2}g_2(\eta), \tag{27}$$

and obtain:

$$g_2'' - \eta g_2' + g_2 = A_1^2(\bar{u}\bar{\rho})^{-1}(D_2\eta^2 - D_1)e^{-\frac{1}{2}\eta^2}. \tag{27a}$$

One complementary function of this equation is η. Hence put:

$$g_2 = \eta h_2(\eta), \qquad \text{or} \qquad w_2 = x^{-1}\eta e^{-\frac{1}{2}\eta^2}h_2, \tag{28}$$

and obtain:

$$\eta h_2'' + (2 - \eta^2)h_2' = A_1^2(\bar{u}\bar{\rho})^{-1}(D_2\eta^2 - D_1)e^{-\frac{1}{2}\eta^2}, \tag{28a}$$

or

$$\frac{d}{d\eta}(\eta^2 e^{-\frac{1}{2}\eta^2}h_2') = A_1^2(\bar{u}\bar{\rho})^{-1}(D_2\eta^2 - D_1)\eta e^{-\eta^2}. \tag{28b}$$

Performing the integration gives:

$$h_2' = \tfrac{1}{2}A_1^2(\bar{u}\bar{\rho})^{-1}e^{-\frac{1}{2}\eta^2}[D_1\eta^{-2} - D_2(1 + \eta^{-2})] + C\eta^{-2}e^{\frac{1}{2}\eta^2}, \tag{29}$$

where C is a constant of integration. In order that w_2 vanish exponentially when η tends to infinity, C must be zero. The second integration gives:

$$h_2 = \tfrac{1}{2}A_1^2(\bar{u}\bar{\rho})^{-1}\left[(D_2 - D_1)\eta^{-1}e^{-\frac{1}{2}\eta^2} - \left(\frac{\pi}{2}\right)^{\frac{1}{2}} D_1 \operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right)\right] + D, \quad (29a)$$

where D is a constant of integration and other values are obtained by a partial integration. Also:

$$\int_0^{\eta} e^{-\frac{1}{2}\eta^2}d\eta = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right). \quad (29b)$$

In order that w_2 should be an even function of η, h_2 must be an odd function and D must be equal to zero. Therefore one obtains:

$$w_2 = \tfrac{1}{2}A_1^2(\bar{u}\bar{\rho})^{-1}x^{-1}\left[(D_2 - D_1)e^{-\eta^2} - \left(\frac{\pi}{2}\right)^{\frac{1}{2}} D_1\eta e^{-\frac{1}{2}\eta^2} \operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right)\right]. \quad (29c)$$

Application of this procedure to the energy equation, preserving only the terms of order x^{-2}, and calculation of all the derivatives, gives the following result:

$$\frac{\partial^2 z_2}{\partial y^2} - \bar{\alpha}\frac{\partial z_2}{\partial x} = \tfrac{1}{2}(3\bar{k}\bar{T})^{-1}x^{-2}F(\eta), \quad (30)$$

where the symbols used denote:

$$\bar{\alpha} = \frac{\bar{u}\bar{\rho}c_v}{\bar{k}}. \quad (30a)$$

The dimension of $\bar{\alpha}$ is length^{-1}.

$$F(\eta) = A_1^2\bar{u}\left[(K_1 - K_2\eta^2)e^{-\frac{1}{2}\eta^2} + K_3\eta(\eta^2 - 3) \operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right)\right]e^{-\frac{1}{2}\eta^2}, \quad (30b)$$

$$K_1 = C[\mu_0\bar{\mu}^{-1}(a_1 + b_1) + \alpha]\bar{p} + C^2[\bar{p} - b_1 3\bar{T}c_p\mu_0\bar{\nu}^{-1}] - \bar{p}, \quad (30c)$$

$$K_2 = \tfrac{1}{2}[C(a_1 + b_1)\mu_0\bar{\mu}^{-1}]\bar{p} + C^2[(\alpha + 1)\bar{p} - 2b_1 3\bar{T}c_p\mu_0\bar{\nu}^{-1}] - a_1\mu_0\bar{\mu}^{-1}\bar{u}^2\bar{\rho}, \quad (30d)$$

$$K_3 = \tfrac{1}{2}[1 - C(a_1 + b_1)\mu_0\bar{\mu}^{-1}]. \quad (30e)$$

The dimension of a coefficient K_i is lb./sq.in. Put:

$$\bar{\eta} = y\left(\frac{\bar{u}\bar{\rho}c_v}{2\bar{k}x}\right)^{\frac{1}{2}} = \eta\delta^{-\frac{1}{2}}, \qquad \delta = \bar{k}(\bar{\mu}c_v)^{-1}. \quad (31)$$

$$z_2 = x^{-1}f_2(\bar{\eta}). \quad (31a)$$

The coefficient δ is a dimensionless one. Obtain:

$$f_2'' + \bar{\eta} f_2' + 2f_2 = \beta F(\bar{\eta}), \tag{32}$$

$$\beta = (3\bar{T}\bar{u}\bar{\rho}c_v)^{-1}. \tag{32a}$$

The dimension of β is inch-sec./lb. Put:

$$f_2 = e^{-\frac{1}{4}\bar{\eta}^2} g_2(\bar{\eta}), \tag{33}$$

and obtain:

$$g_2'' - \bar{\eta} g_2' + g_2 = \beta F(\bar{\eta}) e^{\frac{1}{4}\bar{\eta}^2}. \tag{33a}$$

One complementary function of this equation is $\bar{\eta}$. Hence put:

$$g_2 = \bar{\eta} h_2(\bar{\eta}), \qquad z_2 = x^{-1}\bar{\eta} e^{-\frac{1}{4}\bar{\eta}^2} h_2, \tag{34}$$

to obtain:

$$\bar{\eta} h_2'' + (2 - \bar{\eta}^2) h_2' = \beta F(\bar{\eta}) e^{\frac{1}{4}\bar{\eta}^2}, \tag{34a}$$

or

$$\frac{d}{d\eta}(\bar{\eta}^2 e^{-\frac{1}{4}\bar{\eta}^2} h_2') = \beta \bar{\eta} F(\bar{\eta}). \tag{34b}$$

Hence:

$$h_2' = \beta \bar{\eta}^{-2} e^{\frac{1}{4}\bar{\eta}^2} \int_0^{\bar{\eta}} \bar{\eta} F(\bar{\eta}) d\bar{\eta} + C. \tag{35}$$

In order that z_2 should vanish exponentially when η tends to infinity, C must be equal to zero. Applying integrations by parts to calculate the integral on the right hand side of eq. (35), gives the following solution:

$$h_2' = \beta \bar{u} A_1^2 \left\{ \left[(b_1 \bar{\eta}^{-2} + b_2) + (b_3 \bar{\eta}^{-1} - b_4 \bar{\eta}) \operatorname{erf}\left(\frac{\bar{\eta}\delta^{\frac{1}{2}}}{2^{\frac{1}{2}}} \right) \right] e^{(\frac{1}{4} - \delta)\bar{\eta}^2} \right.$$
$$\left. - b_3 \bar{\eta}^{-2} e^{\frac{1}{4}\bar{\eta}^2} \int_0^{\bar{\eta}} e^{-(\delta/2)\bar{\eta}^2} \operatorname{erf}\left(\frac{\bar{\eta}\delta^{\frac{1}{2}}}{2^{\frac{1}{2}}} \right) d\bar{\eta} \right\}, \tag{36}$$

where the values of the coefficients are:

$$b_1 = \tfrac{1}{2}\delta^{-1}(K_2 - K_1) + (2\pi)^{-\frac{1}{2}} K_3 \left(\frac{3}{\delta} - 4 \right), \tag{36a}$$

$$b_2 = \tfrac{1}{2} K_2 - (2\pi)^{-\frac{1}{2}} K_3 \delta, \tag{36b}$$

$$b_3 = 3K_3(\delta^{-\frac{1}{2}} - \delta^{\frac{1}{2}}), \tag{36c}$$

$$b_4 = K_3 \delta^{\frac{1}{2}}, \tag{36d}$$

$$\operatorname{erf}\left(\frac{\bar{\eta}\delta^{\frac{1}{2}}}{2^{\frac{1}{2}}} \right) = \left(\frac{2\delta}{\pi} \right)^{\frac{1}{2}} \int_0^{\bar{\eta}} e^{-(\delta/2)\bar{\eta}^2} d\bar{\eta}. \tag{36e}$$

To find the value of h_2 again integrations by parts were used. In order that z_2 should be an even function of η, h_2 must be an odd function of η and the constant of integration must vanish. Changing $\bar{\eta}$ into η gives the following expression containing four integrals which were not solved: *

$$z_2 = \beta A_1^2 \bar{u} x^{-1} \left\{ C_1 e^{-\delta\eta^2} + C_2 \eta e^{-[1/(2\delta)]\eta^2} \operatorname{erf}\left[\left(1 - \frac{1}{(2\delta)} \right)^{\frac{1}{2}} \eta \right] + C_3 \eta e^{-\frac{1}{2}\eta^2} \operatorname{erf}\left(\frac{\eta}{\sqrt{2}} \right) + C_4 [\eta e^{-[1/(2\delta)]\eta^2} (I_a - I_b) + I_c] - C_5 \eta e^{-[1/(2\delta)]\eta^2} \left[\left(\frac{\pi\delta}{2} \right)^{\frac{1}{2}} \operatorname{erf}\left(\frac{\eta}{2^{\frac{1}{2}}\delta^{\frac{1}{2}}} \right) \cdot I_c + I_d \right] \right\} = \beta A_1^2 \bar{u} x^{-1} G(\eta), \quad (37)$$

$$\operatorname{erf}\left[\left(1 - \frac{1}{(2\delta)} \right)^{\frac{1}{2}} \eta \right] = (2\delta - 1) \left(\frac{2}{\pi\delta} \right)^{\frac{1}{2}} \int_0^\eta e^{-(1-[1/(2\delta)])\eta^2} d\eta, \quad (37a)$$

$$I_a = \int_0^\eta \eta^{-1} e^{-\frac{1}{2}(1+\delta^{-1})\eta^2} \operatorname{erf}\left(\frac{\eta}{\sqrt{2}} \right) d\eta, \quad (37b)$$

$$I_b = \int_0^\eta \eta^{-1} e^{-\frac{1}{2}(1-\delta^{-1})\eta^2} \operatorname{erf}\left(\frac{\eta}{\sqrt{2}} \right) d\eta, \quad (37c)$$

$$I_c = \int_0^\eta e^{-\frac{1}{2}\eta^2} \operatorname{erf}\left(\frac{\eta}{\sqrt{2}} \right) d\eta, \quad (37d)$$

$$I_d = \int_0^\eta \left[e^{-\frac{1}{2}\eta^2} \operatorname{erf}\left(\frac{\eta}{\sqrt{2}} \right) \int_0^\eta e^{(2\delta)^{-1}\eta^2} d\eta \right] d\eta, \quad (37e)$$

$$\operatorname{erf}\left(\frac{\eta}{2^{\frac{1}{2}}\delta^{\frac{1}{2}}} \right) = \left(\frac{2}{\pi\delta} \right)^{\frac{1}{2}} \int_0^\eta e^{-(2\delta)^{-1}\eta^2} d\eta, \quad (37f)$$

$$C_1 = -b_1, \qquad C_2 = \left(\frac{\pi}{2\delta} \right)^{\frac{1}{2}} [b_2(2\delta - 1)^{-1} - C_1] - K_3\delta^{\frac{1}{2}}(\delta - 1)^{-1}(2\delta - 1)^{-1},$$
$$C_3 = K_3(\delta - 1)^{-1}, \qquad C_4 = 3K_3(\delta - 1), \qquad C_5 = C_4\delta^{-1}. \quad (37g)$$

Application of this procedure to the equation of continuity with $v = v_1 + v_2$ gives:

$$\frac{\partial v_2}{\partial y} = \bar{u} \left(\frac{\partial w_2}{\partial x} - z_1 \frac{\partial w_1}{\partial x} - \frac{\partial z_2}{\partial x} + w_1 \frac{\partial z_1}{\partial x} \right) + z_1 \frac{\partial v_1}{\partial y} - v_1 \frac{\partial z_1}{\partial y}. \quad (38)$$

* As is known the error function is tabulated (see: A. Fletcher, F. C. P. Miller, L. Rosenhead, "An Index of Mathematical Tables." McGraw-Hill Co., New York, 1946) but the author did not succeed in finding the tables of more complicated functions. The unsolved integrals may be evaluated in each particular case by numerical methods.

With the use of eqs. (24) and (30) one can easily transform (38) into:

$$\frac{\partial v_2}{\partial y} = \bar{\nu}\left\{\frac{\partial^2 w_2}{\partial y^2} + C\left[(1-C)w_1\frac{\partial^2 w_1}{\partial y^2} - (1-C)\left(\frac{\partial w_1}{\partial y}\right)^2\right]\right\}$$
$$- \bar{k}(c_v\bar{\rho})^{-1}\frac{\partial^2 z_2}{\partial y^2} + \tfrac{1}{2}A_1^2\bar{\rho}^{-1}x^{-2}(D_1 - D_2\eta^2)e^{-\eta^2} + \tfrac{1}{2}\bar{R}(\mathfrak{J}c_v\bar{\rho})^{-1}x^{-2}F(\eta). \quad (38a)$$

Integrating term by term and applying integrations by parts to particular integrals gives the solution:

$$v_2 = A_1^2 x^{-\frac{1}{2}}\left[C_9(1-\eta^2)e^{-\frac{1}{2}\eta^2}\operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right) - C_{10}\eta e^{-\eta^2} - C_{11}\operatorname{erf}\eta - C_{12}I_t\right]. \quad (39)$$

Since v_2 must vanish for $\eta = y = 0$, the constant of integration must be equal to zero. The symbols used denote:

$$I_t = (C_2 - C_7\eta^2)e^{-[1/(2\delta)]\eta^2}\operatorname{erf}\left[\left(1 - \frac{1}{(2\delta)}\right)^{\frac{1}{2}}\eta\right] + (C_6e^{-\delta\eta^2} + C_8e^{-\eta^2})\eta$$
$$+ C_3(1-\eta^2)e^{-\frac{1}{2}\eta^2}\operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right) + (C_4I_1 - C_5I_2)e^{-[1/(2\delta)]\eta^2}(1 - \delta^{-1}\eta^2)$$
$$+ (C_4e^{-(1/\delta)\eta^2} - C_5\eta e^{-[1/(2\delta)]\eta^2}I_3)e^{-\frac{1}{2}\eta^2}\operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right) + C_5\eta e^{-(1/\delta)\eta^2}I_4, \quad (40)$$

$$I_1 = I_a - I_b, \qquad I_2 = \left(\frac{\pi\delta}{2}\right)^{\frac{1}{2}}\operatorname{erf}\left(\frac{\eta}{2^{\frac{1}{2}}\delta^{\frac{1}{2}}}\right)\cdot I_c + I_d,$$

$$I_3 = \int_0^\eta e^{[1/(2\delta)]\eta^2}d\eta + \left(\frac{\pi\delta}{2}\right)^{\frac{1}{2}}\operatorname{erf}\left(\frac{\eta}{2^{\frac{1}{2}}\delta^{\frac{1}{2}}}\right), \qquad I_4 = I_c,$$

$$C_6 = -2\delta C_1, \quad C_7 = \delta^{-1}C_2, \quad C_8 = (2\delta - 1)\left(\frac{2}{\pi\delta}\right)^{\frac{1}{2}}C_2 + \left(\frac{2}{\pi}\right)^{\frac{1}{2}}C_3, \quad (40a)$$

$$C_9 = 2^{-\frac{1}{2}}\bar{\nu}^{\frac{1}{2}}[\bar{R}(\mathfrak{J}c_v\bar{\rho})^{-1}\bar{u}^{\frac{1}{2}}K_3 - 2^{-\frac{1}{2}}(\pi\bar{u}^{-1})^{\frac{1}{2}}\bar{\rho}^{-1}D_1], \quad (40b)$$

$$C_{10} = (2^{-1}\bar{\nu})^{\frac{1}{2}}\{C(C-1)\bar{u}^{\frac{1}{2}} - (2\bar{u}^{\frac{1}{2}}\bar{\rho})^{-1}D_1 - \bar{R}(\mathfrak{J}c_v\bar{\rho})^{-1}\bar{u}^{\frac{1}{2}}[2^{-1}K_2 - (2\pi)^{-\frac{1}{2}}K_3]\}, \quad (40c)$$

$$C_{11} = -(2^{-1}\bar{\nu})^{\frac{1}{2}}\{(2\bar{\rho})^{-1}(\pi\bar{u}^{-1})^{\frac{1}{2}}D_1 + 2^{-1}\pi^{\frac{1}{2}}[2C(C-1)\bar{u}^{\frac{1}{2}} - D_3\bar{\rho}^{-1}\bar{u}^{-\frac{1}{2}}] + \bar{R}(\mathfrak{J}c_v\bar{\rho})^{-1}\bar{u}^{\frac{1}{2}}[(K_1 - \tfrac{1}{2}K_2)\tfrac{1}{2}\pi^{\frac{1}{2}} - 2^{-\frac{1}{2}}K_3]\}, \quad (40d)$$

$$C_{12} = (2\bar{\nu})^{-\frac{1}{2}}\bar{k}(c_v\bar{\rho})^{-1}\bar{u}^{\frac{1}{2}}\beta. \quad (40e)$$

For $\eta = 0$, $v_2 = 0$, but for $\eta \to \infty$ one obtains $v_{2,\,\eta\to\infty} \to -C_{11}$. Hence the second approximation of v remains finite as η tends to infinity. This result is similar to that one in Goldstein's work. It is a general feature

of solutions of boundary layer equations by this method that v remains finite as η or y tend to infinity.*

One can easily calculate μ_2 and k_2 using the value T_2. Substitution of all the second approximations into the second eq. of motion and keeping only the terms of order x^{-2} gives the result:

$$\bar{R}\bar{T}(1 - z_1 - z_2)\frac{\partial p_2}{\partial y} + p_2\bar{u}\frac{\partial v_1}{\partial x} + \bar{R}\bar{T}\left[(\bar{\mu} - \tfrac{2}{3}\mu_0)\bar{\mu}\frac{\partial^2 w_1}{\partial x \partial y} - \tfrac{4}{3}\mu_0\frac{\partial^2 v_1}{\partial y^2}\right] = 0, \quad (41)$$

where $p_2 = p$. Put $p_2 = \bar{p}(1 - t_2)$ and assume that t_2 is of the order x^{-1}. Preserving only the terms of order x^{-2} one obtains:

$$\bar{R}\bar{T}\bar{p}\frac{\partial t_2}{\partial y} = \bar{p}\bar{u}\frac{\partial v_1}{\partial x} + \frac{\partial}{\partial y}\left\{(\bar{\mu} - \tfrac{2}{3}\mu_0)\bar{u}\frac{\partial w_1}{\partial x} - \tfrac{4}{3}\mu_0\frac{\partial v_1}{\partial y}\right\}\bar{R}\bar{T}. \quad (41a)$$

After some simple transformations and integrations one has:

$$t_2 = x^{-1}e^{-\frac{1}{4}\eta^2}(C_{13}\eta^2 + C_{14}) + \text{const.} \quad (41b)$$

Since for $\eta \to \infty$, t_2 must tend to zero, the constant of integration must vanish.

$$C_{13} = \tfrac{1}{2}A_1\frac{\bar{u}}{\bar{p}}[(2 - C)\bar{\mu} + 2(\tfrac{2}{3}C - 1)\mu_0], \quad (41c)$$

$$C_{14} = \tfrac{1}{2}A_1\frac{\bar{u}}{\bar{p}}[2(1 - \tfrac{2}{3}C)\mu_0 - \bar{\mu}]. \quad (41d)$$

Dimension of C_{ij} is inch $^{\frac{1}{2}}$.

IV. Third approximation.

Put into the first equation of motion the values $u_3 = \bar{u}(1 - w_1 - w_2 - w_3)$, $v = v_1 + v_2$, p_2, T_2 and μ_2. w_3 is of order $x^{-\frac{1}{2}}$ and in the equation only the terms of order $x^{-5/2}$ should be kept. After all the necessary transformations and calculation of the derivatives, the result may be given in the form:

$$\bar{R}\bar{T}\bar{\mu}\frac{\partial^2 w_3}{\partial y^2} - \bar{p}\bar{u}\frac{\partial w_3}{\partial x} = \tfrac{1}{2}x^{-5/2}e^{-\frac{1}{4}\eta^2}F(\eta), \quad (42)$$

$$F(\eta) = \sum_{i=1}^{7} F_i(\eta), \quad (42a)$$

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* Goldstein, *loc. cit.*

$$F_1(\eta) = \tfrac{1}{2}A_1{}^3\bar{R}\bar{T}(Cb_2\mu_0\bar{\mu}^{-1} - 1)\left\{\left(\frac{\pi}{2}\right)^{\frac{1}{2}} D_1\eta e^{-\frac{1}{2}\eta^2}\operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right)(3 - \eta^2)\right.$$
$$\left. + (2D_2 - D_1)\eta^2 e^{-\eta^2}\right\} + A_1{}^3\bar{R}\bar{T}e^{-\eta^2}(D_2\eta^2 - D_1), \quad (42b)$$

$$F_2(\eta) = A_1{}^3\bar{R}\bar{T}\left\{ b_2\bar{u}^2\mu_0\bar{\nu}^{-1}\beta G(\eta)\right.$$
$$\left. - \tfrac{1}{2}\left[(D_2 - D_1)e^{-\eta^2} - \left(\frac{\pi}{2}\right)^{\frac{1}{2}} D_1\eta e^{-\frac{1}{2}\eta^2}\operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right)\right]\right\}(\eta^2 - 1), \quad (42c)$$

$$F_3(\eta) = A_1{}^3\bar{R}\bar{T}\left\{ - 2^{\frac{1}{2}}\bar{\rho}\left(\frac{\bar{u}}{\bar{\nu}}\right)^{\frac{1}{2}}\left[C_9(1 - \eta^2)e^{-\frac{1}{2}\eta^2}\operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right)\right.\right.$$
$$\left. - C_{10}\eta e^{-\eta^2} - C_{11}\operatorname{erf}\eta - C_{12}I_t\right] - \tfrac{1}{2}b_2C\mu_0\bar{\mu}^{-1}\left[(D_1 - 2D_2)\eta e^{-\eta^2}\right.$$
$$\left.\left. - \left(\frac{\pi}{2}\right)^{\frac{1}{2}} D_1e^{-\frac{1}{2}\eta^2}\operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right)(1 - \eta^2)\right]\right\}\eta, \quad (42d)$$

$$F_4(\eta) = \tfrac{4}{3}A_1\bar{R}\bar{T}\bar{\mu}(\tfrac{1}{2}\eta^4 + 3\eta^2 - \tfrac{3}{2}), \quad (42e)$$

$$F_5(\eta) = \tfrac{1}{6}A_1\bar{R}\bar{T}\bar{\mu}(1 - C)(\eta^3 - 3\eta^2 - 2\eta + 3), \quad (42f)$$

$$F_6(\eta) = \tfrac{1}{2}A_1{}^3\bar{R}\bar{T}(C - 1)\left[(D_1 - 2D_2)\eta e^{-\eta^2}\right.$$
$$\left. - \left(\frac{\pi}{2}\right)^{\frac{1}{2}} D_1e^{-\frac{1}{2}\eta^2}\operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right)(1 - \eta^2)\right] - b_2\bar{u}^2\mu_0\bar{\nu}^{-1}\beta G'(\eta)\Big\}\eta. \quad (42g)$$

The derivative G' is to be taken with respect to η.

$$F_7(\eta) = \bar{u}^{-1}\bar{p}\bar{R}\bar{T}[C_{13}(\eta^2 - 2) + C_{14}]\eta^2. \quad (42h)$$

Each function $F_i(\eta)$ has the dimension lb. in.$^{\frac{1}{2}}$ sec.$^{-1}$. Put:

$$w_3 = x^{-\frac{1}{2}}f_3(\eta). \quad (43)$$

Then:

$$\bar{u}\bar{p}(f_3'' + \eta f_3' + 3f_3) = e^{-\frac{1}{2}\eta^2}F(\eta). \quad (43a)$$

Put:

$$f_3 = e^{-\frac{1}{2}\eta^2}g_3(\eta). \quad (44)$$

Then:

$$\bar{u}\bar{p}(g_3'' - \eta g_3' + 2g_3) = F(\eta). \quad (44a)$$

One complementary function of this equation is $(1 - \eta^2)$. Therefore put:

$$g_3 = (1 - \eta^2)h_3, \quad (45)$$

$$w_3 = x^{-\frac{1}{2}}e^{-\frac{1}{2}\eta^2}(1 - \eta^2)h_3. \quad (45a)$$

Then:

$$\bar{u}\bar{p}[(1-\eta^2)h_3'' - 5\eta h_3' + \eta^3 h_3'] = F(\eta), \tag{45b}$$

or

$$\bar{u}\bar{p}\frac{d}{d\eta}[(1-\eta^2)^2 e^{-\frac{1}{2}\eta^2} h_3'] = (1-\eta^2)e^{-\frac{1}{2}\eta^2}F(\eta). \tag{46}$$

Finally:

$$\bar{u}\bar{p}(1-\eta^2)^2 e^{-\frac{1}{2}\eta^2} h_3' = \int_0^\eta (1-\eta^2)e^{-\frac{1}{2}\eta^2}F(\eta)d\eta + C. \tag{47}$$

The derivative $\partial w_3/\partial\eta$ must vanish when η vanishes, and therefore $(1-\eta^2)h_3'$ must vanish when η vanishes. Hence $C = 0$. When the upper limit becomes infinite, the integral on the right is finite since it is composed of integrals of the form

$$\int_0^\infty e^{-\frac{1}{2}\eta^2}d\eta, \qquad \int_0^\infty e^{-\frac{1}{2}\eta^2}d\eta,$$

which are finite. Thus the third approximation is not successful. A similar result was obtained in Goldstein's work.*

* The author expresses his thanks to Dr. P. W. Ketchum, Professor of Mathematics, University of Illinois, for the reading of the manuscript.

CHEMICAL COMPOSITION OF SEMINAL VESICLE CALCULI.

BY

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Calculi removed by perineal operation from the seminal vesicle of a patient, J. K., by Dr. Horace L. Weinstock, had a total weight of 1.6822 grams, and consisted of six (6) fragments. The individual calculi weighed from 0.0504 to 0.6962 gram. The calculi were ground to an impalpable powder in an agate mortar, and subjected to preliminary qualitative tests by the procedure of Looney (1) for urinary calculi. These tests, plus confirmatory tests, showed the presence of calcium, magnesium, phosphate anion, and protein. Uric acid, urates, carbonates, oxalates, cystine, xanthine, fibrin, urostealith, and the ammonium cation were absent. The benzidine and guaiac tests for the presence of occult blood were negative.

Calcium and magnesium were determined quantitatively by the procedure of McCrudden (2) on a sample weighing 0.2000 gram and completely dissolved in hot dilute hydrochloric acid. The calcium oxide obtained weighed 0.0890 gram, the magnesium pyrophosphate obtained weighed 0.0038 gram. Accordingly, the calculi contained 44.50 per cent. calcium oxide and 0.344 per cent. magnesium oxide.

Phosphates were determined on another sample, also weighing 0.2000 gram. The organic matter was oxidized by boiling with concentrated sulfuric acid and the addition of potassium nitrate. The phosphoric acid was isolated as ammonium phosphomolybdate, converted into magnesium ammonium orthophosphate, and finally weighed as magnesium pyrophosphate, obtaining 0.1290 gram of the latter compound (3). Accordingly, the calculi contained 41.15 per cent. phosphoric anhydride.

From the respective per cents. of calcium oxide, magnesium oxide, and phosphoric anhydride, it may readily be calculated that the calculi contained:

Tricalcium phosphate, $Ca_3(PO_4)_2$	68.23%
Dicalcium phosphate, $CaHPO_4.2H_2O$	23.04%
Trimagnesium phosphate, $Mg_3(PO_4)_2$	0.40%

The total nitrogen content of a sample weighing 0.0998 gram was determined by the micro-Kjeldahl method of Henwood and Garey (4)

using 0.02 normal solutions of sulfuric acid and sodium hydroxide for the residual titration. From this titration, the protein content was calculated, and found to be 8.36 per cent.

The calculi accordingly consisted of secondary and tertiary phosphates of calcium, tertiary phosphate of magnesium, and protein, and had the following percentage composition:

$Ca_3(PO_4)_2$	68.23
$CaHPO_4.2H_2O$	23.04
$Mg_3(PO_4)_2$	0.40
Protein	8.36
Total	100.03

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NOTES FROM THE NATIONAL BUREAU OF STANDARDS.*

GUIDED MISSILES.

The *Bat*, first fully automatic guided missile successfully used in combat by any nation, offers evidence that guided missiles have passed from the experimental stage into practical application as a combat weapon. One of several guided missiles developed by the National Bureau of Standards under the sponsorship of the Navy Bureau of Ordnance, the *Bat* has led to further research on advanced forms, the "Kingfisher Series."

Work on guided missiles constituted one of the major war activities of the National Bureau of Standards. During the course of the work, such missiles as the *Robin*, the *Pelican*, and the *Bat* were brought to a state of completed development. Because of the much greater operational flexibility of the *Bat*, however, it was the only one of these missiles carried into large-scale production and combat use. The program on guided missiles began in the fall of 1940, when the National Defense Research Committee requested the Bureau to permit the appointment of Dr. Hugh L. Dryden as aerodynamics consultant in connection with a contract for development of a glide bomb capable of being directed to the target by radio remote control utilizing television sighting.

ROBIN AND PELICAN.

In April 1942, the Bureau acquired the use of an area for flight tests near Warren Grove, N. J., where tests were made with aircraft supplied by the Navy. From this period on, the several projects designated *Robin*, *Pelican*, and *Bat* were prosecuted simultaneously. Each of these projects utilized a glider in three sizes, designed to be carried externally by the dropping airplane. The unique type of aerodynamic control, suitable for homing missiles, was developed in the Bureau's Aerodynamics Laboratory.

Project *Robin* contemplated a missile incorporating a television transmitter with pickup tube in the nose of the bomb and manual remote radio control by an operator with a television receiver in front of him. The full-scale missile was to carry a standard M-34 2,000-pound bomb as pay load. Twenty-six model gliders and fourteen full-scale gliders were flight-tested in connection with this project. Twenty-four of the flights were with television equipment. Gyro-stabilization was introduced in the seventh test. The tests were concluded in July 1943,

* Communicated by the Director.

when radar homing appeared to be a more practical solution of the guided missile problem.

The *Pelican* was a radar device, but the missile contained only a receiver; the transmitter was on the "mother" airplane. A pulsed radar system was used, the transmitter emitting short pulses of high intensity. Directional information obtained from the returning echoes activated mechanisms within the missile which kept it headed for the target. The first flight test of a missile carrying an experimental radar receiver was made at the Warren Grove test area on November 12, 1942, followed a month later by the first flight showing homing control, the transmitter in both cases being located on the ground.

Various difficulties, both aerodynamic and radar, as well as development of an improved target selector circuit, delayed final selection of a production design. The first production glider was tested against an elevated beacon on June 10, 1943. Attention was then turned to tests against a water target and to illuminating techniques with the transmitter in an airplane, all previous tests having been made with transmitter on the ground. Tests of this nature were made against a barge in the Chesapeake Bay in October 1943. In 1944, however, the *Pelican* was returned to an experimental status, largely because the *Bat* appeared to be nearly ready for use.

THE BAT.

Designed to glide silently at the speed of a fighter plane, with a 1,000-lb. bomb as cargo, the *Bat* is carried under the wing or fuselage of a Navy fighter or bomber and may be released at a distance as great as 14 miles from the target. Once the crew of the mother plane has located a ship or other target, the missile is set for this particular target and the release button is pushed. From then on the *Bat* automatically follows every maneuver of the target until it eventually strikes home, while the mother plane is free to proceed as its pilot wishes.

The code name *Bat* suggests the principle upon which this guided missile operates. A live bat gives out short pulses of sound and is guided by the echo, thus avoiding collision in the dark. The *Bat* missile emits pulsed microwave electromagnetic radiation and is directed by the radar echoes from the target. The radar robot pilot can, of course, "see" the target under any conditions of visibility.

In normal usage the target is first located by the standard search radar of the mother plane. The plane is then headed toward the target, and the radar transmitter and receiver in the *Bat* are aimed in its direction. Target data from the radar in the missile are displayed on a separate indicator in the mother plane under the control of the operator. After the radar equipment is manually adjusted to the proper conditions, it is switched to automatic tracking, and the missile is released. Echoes from the target are continuously detected by the radar receiver in the

missile, and the output supplies corrective signals to the flight control units so as to direct the missile toward the target.

The outstanding features of the *Bat* as a combat weapon are (1) long range, which provides the ability to attack heavily armed targets with accuracy from outside the antiaircraft range; (2) improved accuracy, a result of the self-homing feature; (3) low angle of flight, which reduces high altitude problems; (4) high pay load; and (5) self-guidance after release, permitting the carrying plane to maneuver as desired.

The *Bat* is equipped with a gyroscopic stabilizing unit and a servo system to move the control surfaces of the air stabilizer. This equipment, together with the bomb load, is mounted in a glider type airframe about 12 feet long with a 10-foot wing span.

The preliminary design of a glider of 10-foot span was started in June 1943, and construction of the first glider began approximately three months later. The Navy had placed a contract with Bell Telephone Laboratories for the design and development of the radar homing system early that year. In October these designs were crystallized, and in January 1944 one of the first three 10-foot gliders was sent to Bell Laboratories. The first flight test of a 10-foot glider with radio remote control was made on May 10, 1944. Seven tests were made during the next few months with the *Pelican* receiver. As a result of these tests and of wind tunnel tests, the design was slightly modified. Between October 7 and 19, 1944, a series of 11 tests with *Bat* equipment were made against the target ship *James Longstreet* with excellent results.

Development of this complex weapon resulted from close cooperation between a number of organizations. Dr. Hugh L. Dryden, of the National Bureau of Standards, coordinated the technical work of the civilian agencies, while Captain Dundas P. Tucker, of the Bureau of Ordnance, integrated the activities of the military groups. According to the Navy, the *Bat* was very effective against Japanese combatant and merchant shipping during the last year of the war.

THERMAL SHOCK FAILURE OF ENAMELWARE UTENSILS.

A series of tests recently completed at the National Bureau of Standards have indicated the mechanism that causes enamel to chip off cooking utensils when subjected to stresses due to sudden heating and cooling. The work, which supplements investigations made by the enamelware industry over a period of years, was conducted by W. N. Harrison and J. C. Richmond, of the Enameled Metals Section, at the request of the Procurement Division of the Treasury Department.

Thermal shock failure, or chipping, of an enamelware pan characteristically occurs after the pan has been allowed to boil dry and cold water has been poured into it to counteract the excessive heat. This brings

about the severe shock which, if repeated often enough, may cause failure of the enamel, and unless the pan is of good quality, may do so quickly.

In one group of experiments, using two identical sets of 2-quart pudding pans, the vessels were heated first, then one set quenched quickly with water and the other allowed to cool before water was poured in. In the first case, the pans failed at an average of 7 cycles, that is after quenching 7 times from successively higher temperatures, but in the second instance none of them chipped in 14 cycles, which was as far as the test was carried. These results demonstrated that the sudden quenching is the part of the cycle that starts the failure, although it may not show up before reheating. The explanation, given by Bureau investigators, is that quenching makes the surface contract faster than the underlying material, causing cracks which may be microscopic. The quenching water enters these cracks and, when the pan is reheated, steam is formed which blows the enamel off.

Other experiments confirmed previous findings that a thin coat of enamel stands thermal shock better than a thick one. No consistent effect due to differences in thickness of the metal base was noted.

In the early test, a hot plate was used and satisfactory results obtained when 2-quart pudding pans were tested. It was thought desirable, however, to use a method which would heat the largest as well as the smallest utensils uniformly, so a thermostatically controlled gas fired oven was developed in which the utensils were heated to predetermined temperatures. Ice water was then poured into them while they were still in the oven. This procedure was repeated at successively higher temperatures until a failure occurred, or until a stated series of quenchings had been completed without damage.

Specimens for the work were furnished by the Enameled Utensil Manufacturer's Council. It is expected that these experiments, by contributing to a better understanding of the mechanism which causes thermal shock failure, will be of assistance in the production and procurement of improved porcelain enameled utensils.

NEW ROTARY CONCENTRIC-TUBE DISTILLING COLUMN.

A new rotary concentric-tube distilling column, having a very high efficiency factor for fractional distillation, has been developed at the National Bureau of Standards by Charles B. Willingham, Vincent A. Sedlak, James W. Westhaver, and Frederick D. Ressini, in connection with the cooperative research program on hydrocarbons sponsored jointly by the Bureau and the American Petroleum Institute.

The present-day demand for fuels providing greater efficiency, higher speeds, and more concentrated power, as well as the development of various devices making use of new types of hydrocarbons, has resulted in a

transition in the petroleum industry from the manufacture of broad fractions toward the production of narrower fractions and even pure compounds. The experimental program now under way at the National Bureau of Standards has two objectives: (1) the investigation of the chemical constituents of the crude oil based upon the actual isolation of pure hydrocarbons; (2) the securing of high-purity standard samples for the calibration of instruments used in analyzing such complex mixtures as aviation fuels and synthetic rubber components. The new distilling column, which is expected to have wide application to fractionation processes, will aid materially in the Bureau's hydrocarbon research program.

Numerous devices, based on both theory and experiment, are available for producing high separating efficiencies in distilling columns. Among these are open tubes of small diameter, concentric tubes with a small annular space, and parallel plates with a small space between them. All, however, are limited in use by a relatively low throughput, that is, quantity of material that may be volatilized per hour. According to theory, the efficiency may be improved by decreasing the spacing of the tubes or plates of the column, by decreasing the throughput, or by increasing the rate of diffusion of the gas molecules through the distilling column. Decrease in the spacing (or diameter) has already been carried about as far as practicable, while decreasing the throughput below the present low values is not feasible. Moreover, in a static apparatus, for a given temperature and composition, the rate of diffusion of the gaseous molecules is substantially constant.

Accordingly, the new distilling column was designed to improve separating efficiency by increasing the diffusion rate of the molecules in the gas phase. This is accomplished by forcing the gas into turbulence through rotation of the inner closed cylinder in a concentric-tube rectifying section.

The apparatus consists of three parts—the Pyrex head, steel rectifying section, and Pyrex pot. All three sections are provided with external heating elements and copper-constantan thermocouples for temperature regulation.

The steel rectifying section of this distilling column is the empty annular space, 0.043 inch (1.09 mm.) wide, formed by the inside surface of a stationary outer cylinder and the outside surface of a rotating closed inner cylinder, 2.928 inches (7.44 cm.) in outside diameter and 23.0 inches (58.4 cm.) in length. A motor-and-pulley system drives the rotor at speeds up to 4,000 r.p.m. The outer cylinder is enclosed in an asbestos-covered metal heating jacket surrounded by three separate nichrome heating elements for the top, middle, and bottom portions of the jacket, respectively. The heating elements are covered externally with magnesia insulation and a layer of aluminum foil.

The Pyrex glass head consists principally of a water-jacketed condenser directly above and opening into an electrically heated chamber, in which the liquid reflux may be collected and sampled by means of a glass valve. The heated chamber is surrounded by asbestos wool insulation and aluminum foil.

The pot, made of a three-inch Pyrex pipe, is sealed at one end and provided with a butyl carbitol manometer and a tube for withdrawing samples. Heated externally by a Glas-col special sleeve type heater, it is surrounded by magnesia insulation covered with aluminum foil.

Three thermo elements are provided. One measures the difference between the temperature of the top portion of the rectifying section and the liquid-vapor equilibrium in the head. Another measures the difference between the temperature of the middle portion and the mean temperature of the top and bottom portions of the rectifying section. A third is used to determine the difference between the temperature of the bottom of the rectifying section and the temperature of the liquid in the pot.

For high values of throughput—two to four liters of liquid per hour—this distillation column, when operated at 4,000 r.p.m., has an efficiency factor about ten times those previously reported for other rectifying columns. The efficiency factor changes relatively little with throughput at a given speed of rotation, but increases markedly with speed of rotation. This column also has low values of pressure drop per unit throughput, which may be quite advantageous for distillations at low pressure.

THE FRANKLIN INSTITUTE.

STATED MONTHLY MEETING, WEDNESDAY, MAY 21st. 1947.

The President, Mr. Richard T. Nalle, called the meeting to order at 8:15 P.M. and welcomed the guests to the Charles Day Lecture Foundation meeting.

He announced that the minutes of the stated monthly meeting for February were printed in full in the March issue of the JOURNAL OF THE FRANKLIN INSTITUTE, and that if there were no corrections or additions they would stand approved as printed.

The Secretary, Henry B. Allen, then made the following report:

The following members were elected during the month of April.

Sustaining	1
Active	40
Associate	18
Student	20
Total	79

Mr. Nalle stated that the Charles Day Lecture Foundation was established as a memorial to Mr. Charles Day, a good friend of The Franklin Institute, a valued member of the Board of Managers from 1908 to 1931, and Vice President 1920 to 1923. Mr. Day was a distinguished mechanical engineer, lecturer and author of many technical papers on industrial problems of scientific management.

Mr. Nalle then introduced the speaker, Mr. Ralph Kelly, who spoke about "The Effect of the War Upon Production Standards and the Outlook for the Future."

The meeting was adjourned with a rising vote of thanks to the speaker.

HENRY BUTLER ALLEN, *Secretary.*

COMMITTEE ON SCIENCE AND THE ARTS.

(Abstract of Proceedings of Stated Meeting held Wednesday, May 14, 1947.)

HALL OF THE COMMITTEE,
PHILADELPHIA, MAY 14, 1947.

MR. WALTER C. WAGNER IN THE CHAIR.

The following report was presented for final action:

No. 3164: Contribution of Wendell F. Hess to the Art of Welding.

This report recommended the award of a John Price Wetherill Medal to Wendell Frederick Hess, of Troy, New York, "In consideration of his outstanding contributions to the Art and Science of Welding, notably in the field of Electric Resistance Welding."

LIBRARY NOTES.

The Committee on Library desires to add to the collections any technical works that members would wish to contribute. Contributions will be gratefully acknowledged and placed in the library. Duplicates received will be transferred to other libraries as gifts of the donor.

Photostat Service. Photostat prints of any material in the collections can be supplied on request. Orders received in the morning are filled the same day. The average cost for a print 9 × 14 inches is thirty-five cents.

The library and reading room are open on Mondays, Tuesdays, Wednesdays, and Fridays from nine o'clock A.M. until five o'clock P.M., Thursdays from two until ten o'clock P.M., Saturdays from 9 o'clock A.M. until 12 noon as of June 1, 1947.

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NOTES FROM THE BIOCHEMICAL RESEARCH FOUNDATION.

The Blood Picture of Normal Laboratory Animals.[1] **A Review of the Literature 1936–1946.**—MARY VIRGINIA GARDNER.

THE GUINEA PIG.

Guinea pigs are no longer the most commonly used laboratory animal as they probably were before 1930, when Scarborough (1) reviewed the literature on the blood picture of normal guinea pigs. For that reason fewer reports on guinea pig hematology have been appearing in the literature, particularly since 1940.

Method of Obtaining Blood. Blood sufficient for routine hematological examination may be obtained from the marginal ear vein of the guinea pig. When a larger quantity of blood is needed it may be procured by heart puncture.

Red Blood Cells. The erythrocyte of the guinea pig is a nonnucleated biconcave disc which appears orange in Wright-stained preparations. The average of the reported values for the number of erythrocytes in the blood of the adult guinea pig, regardless of sex, is 5,640,000 per cmm. (Table I)

TABLE I.

Erythrocytes.

Age of Guinea Pigs	Sex and No. of Guinea Pigs	Erythrocytes × 10^6/cmm.		Reported by
		Average	Range	
4–12 weeks	8	6.05	4.80–7.42	von Euler (2)
adult	40	5.6	5.14–6.62	Ber (3)
adult	41 M	5.81	3.40–8.81	Botzaris (4)
adult	9 F	5.856	3.97–7.28	Botzaris (4)
adult	23 M	6.10	4.73–7.04	Goodman (5)
adult	3	5.26	5.00–5.60	Gordon (6)
adult	4	5.35	5.20–5.50	Gordon (7)
adult	9	5.06	4.70–5.25	King (8)
adult	16	4.4	4.00–4.90	Sigal (9)
adult	1 F	5.28		Simpson (10)
Average		5.64		

Little daily fluctuation in erythrocyte number over a 3-week period was found by King and Lucas (8) in a study of 9 normal animals.

Considerable spontaneous fluctuation in reticulocyte levels of normal guinea pigs was found by Agren and Caspersson (11) and by Goodman,

[1] Continued from page 435, Vol. 243, No. 5.

Geiger and Klumpp (5). The mean reticulocyte count was found by Barrie (12) to differ with different strains and by von Euler and Malmberg (13) to decrease with age. About 80 per cent. of 1,538 reticulocyte counts on 23 normal male guinea pigs was shown by Goodman, Geiger and Klumpp (5) to range between 0–2.3 per cent. of the total erythrocyte count. The average values reported are shown in Table II.

TABLE II.

Reticulocytes.

Age of Guinea Pigs	Sex and No. of Guinea Pigs	Reticulocytes Per Cent. of Erythrocytes		Reported by
		Average	Range	
4–12 weeks	—		1.0–2.0	von Euler (13)
adult	25		0.61–1.38	Agren (11)
adult	230	0.197		Barrie (12)
adult	115	0.616		Barrie (12)
adult	40	1.2	0.7–2.1	Ber (3)
adult	23 M		0.0–3.8	Goodman (5)
adult	3	0.4	0.3–0.5	Gordon (6)
adult	4	0.17	0.1–0.2	Gordon (7)
adult	1 F	0.4		Simpson (10)
Average		0.423		

Anisocytosis of most guinea pig red blood cells was found by Botzaris (4).

Hemoglobin. A range of 13 to 16 g. hemoglobin per 100 cc. blood (Haden-Hausser hemometer), with an average of 14.7 g. in the blood of 23 normal male guinea pigs, was reported by Goodman, Geiger and Klumpp (5). An average of 87 per cent. hemoglobin (Sahli), with a range of 80 to 96 per cent., was found by Sigal (9) in a study of 16 normal guinea pigs on a diet high in vitamin C.

White Blood Cells. The neutrophil of the guinea pig differs little from that of the human being. The eosinophil of the guinea pig, according to Joyner (14), contains the usual acidophilic granules. These are somewhat smaller than the blue-staining granules of the basophil and are very refractive in wet preparations. Joyner (14) has described the basophil as a dark mass of deeply stained blue granules in the fixed smear. The granules are oval in shape and, in supravitally stained preparations, appear larger and more uniform in size than do those of the eosinophil.

There has been some disagreement among investigators as to the designation of the leukocyte in which the so-called Kurloff body is occasionally seen. Spink (15) reported that the absolute number of Kurloff bodies was dependent upon the monocyte level and also that there appeared to be a numerical relationship between eosinophils and Kurloff

bodies. He described the Kurloff body as a dark red homogeneous mass in the cytoplasm of Wright-stained leukocytes. Ledingham (16) concluded that the cell containing the Kurloff body was "truly a member of the lymphocyte series," and that the Kurloff body was some sort of secretion of the cell. The inclusions were found by Ledingham only occasionally in the blood of fetuses and new-born guinea pigs, but they were easily demonstrated after the 10th to 13th days. Females were found, by him, to have a somewhat higher number of the cells than did males. Joyner (14), using supravitally stained preparations, was able to demonstrate the Kurloff body only in the lymphocyte. The lymphocyte, according to Joyner (14), closely resembles that cell in the blood of the human being; the monocyte, too, resembles the monocyte of the other mammals, having a neutral red rosette of vacuoles in supravital preparations.

The averages of the reported values for the different types of cells, expressed as the number per 100 leukocytes, are:

Neutrophils	40.89 (31–53)
Eosinophils	2.65 (1.6–3.5)
Basophils	0.45 (0.19–0.7)
Lymphocytes	53.81 (39.4–63.6)
Monocytes	2.89 (1.5–8.2)

The average of the reported values for the total number of leukocytes, regardless of sex, is 9,800 (Table III).

An average absolute increase of 3,057 cells per cmm. was found by Menkin, Mattison and Ulled (19) in the leukocyte count of 7 normal guinea pigs over a 5-hour period. The total number and the differential number of leukocytes in the blood of normal guinea pigs were found by King and Lucas (8) to vary widely from day to day.

In a study of 6 new-born guinea pigs, 85 to 90 per cent. of the leukocytes were found, by Ber (3), to be mononuclears.

Platelets. The volume of platelets in 100 cc. blood (Van Allen method) was found by Graña (18) to average 0.70, with a range of 0 to 1.3 in 8 animals studied, and 1.25, with a range of 1.2 to 1.4 in 4 animals studied. The normal platelet count in 1 female guinea pig was found by Simpson, Levy and Cadness (10) to be 380,160 per cmm.

Sedimentation Rate. The average blood sedimentation rate for 90 guinea pigs, Nicolle and Simons (20) found to be 1.06 mm. at the end of 1 hour. The sedimentation had increased to 4.0 mm. at the end of 6 hours and to 8.8 mm. at the end of 24 hours. They found that, although the 4th hour was the best for interpretation, the maximum sedimentation was sometimes not reached for 41 days. They found, too, that the sedimentation rate was only slightly influenced by the source of the blood or the degree of hemorrhage, by differences in diet or sex, by pregnancy, jaundice or repeated bleedings.

TABLE III.
Leukocytes.

Age of Guinea Pigs	Sex and No. of Guinea Pigs	Leukocytes per cmm.		Cells per 100 Leukocytes						Reported by
		Average	Range	Neut.	Eo.	Baso.	Lymph.	Mono.	Kurloff	
4–12 weeks	4	10,140	7,600–11,475							von Euler (2)
adult	40	*ca.* 10,000	6,000–18,000	36.05	3.2	0.49	55.3	2.39	1.43	Ber (3)
adult	41 M	6,472		34.18	1.99	0.57	61.31	1.70	{3–11% of	Botzaris (4)
adult	9 F	6,345		33.54	1.59	0.30	63.60	1.79	{Lymphs.	Botzaris (4)
adult	130	10,138		46.3						Fleisher (17)
adult	6 M	14,130	12,000–18,800							Graña (18)
adult	232	9,688	2,100–44,300	48	4.4		39.4	8.2	2.9	Joyner (14)
adult	9	17,365	13,000–23,700	31.1	3.5	0.19	63.1	1.8		King (8)
adult	39	12,070	3,325–29,300							Menkin (18)
adult	16	7,800	6,100–10,800	45	3.0	0.7	50	1.5		Sigal (9)
adult	1 F	5,700		53			44			Simpson (10)
Average		9,800		40.89	2.65	0.45	53.81	2.89		

Hematocrit. The volume of packed red cells was found by Goodman, Geiger and Klumpp (5) to average 42.0 per cent., with a range of 37 to 47 per cent. in 23 normal male guinea pigs.

No references to blood volume, specific gravity, coagulation time or resistance of red blood cells were found.

The average values recorded here for erythrocyte number, hemoglobin, relative neutrophil number, relative basophil number, total leukocyte count, and platelet number agree within the limits of experimental error with those given by Scarborough (1). The average relative lymphocyte count was found to be somewhat higher, and the relative monocyte and eosinophil count somewhat lower, than those recorded by Scarborough.

(*To be continued*)

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BOOK REVIEWS.

HEATING AND AIR CONDITIONING, by John R. Allen, James Herbert Walker and John William James. Sixth edition. 667 pages, illustrations, plates, 16 × 23 cms. New York, McGraw-Hill Book Company, 1946. Price $5.50.

This book has been familiar to a host of engineers as a text and reference work through the years of much progress in both heating and air conditioning. Now in its sixth edition it has kept up with this progress and today offers a modern treatment in method of presentation obtained by experience, as well as content. The changes in the science of heating and air conditioning since the previous edition (1939) have not been revolutionary so that much of the previous work has been retained. What additional knowledge and improved applications there have been are important as facts and accomplishments which point to the future. The features of this sixth edition are these changes and revisions.

We are undoubtedly on the threshhold of practical use on a much larger scale of the reversed cycle refrigeration system. This interesting subject is given consideration from a theoretical and practical viewpoint. An entirely new section on panel heating is included in the new edition. The chapter on fuels and boilers has been extensively revised; with the inclusion of flue gas efficiency charts for coal, gas, or oil firing. Information relative to odor-removal apparatus and air sterilization has been added. Revisions include the latest rating formulas and design procedures of the furnace heating and residence air conditioning industry. The chapter on hot water heating has been changed to incorporate more up-to-date information on design of small forced circulation systems. Other recent developments in the field of heating, ventilating and air conditioning are covered.

Outstanding in this work is clarity of presentation. The method is that of a text with descriptive matter usually first, then reference to tabular and formula data, and finally an exercise with its progressive steps to solution. Such exercises are constructed from different angles and approaches. Free use is made of photographs and drawings. At the ends of chapters are problems to be solved by the student.

Appendixes contain much valuable information and workable data including the theory of air and water-vapor mixtures for those interested in precise calculations, and a complete understanding of such phenomena. The comprehensive subject index adds much to the value of the work for reference.

R. H. OPPERMANN.

EXPLOSION AND COMBUSTION PROCESS IN GASES. by Wilhelm Jost. First edition. 621 pages, illustration, 16 × 24 cms. New York, McGraw-Hill Book Compay, 1946. Price $7.50.

The importance of knowledge of the physical chemistry of the explosive reaction has been brought to the fore quite suddenly by the war. It was useful in a number of fields and research was stimulated. The literature was found to be in need of a single work complete within wide borders. The book at hand deals with the subject. Originally in German with a preface dated June 1935 it was a classic. The translation to English was undertaken by the publisher and Huber O. Croft at the suggestion of Dr. W. F. Durand and the N.A.C.A., and Dean E. A. Moreland of the N.D.R.C.

The presentation is generally divided into two parts, that referring to the preliminary stages and the origin of an explosion, and that dealing with the phenomena of the completed explosion. Beginning with the spontaneous development of explosions the treatment is rigorous. It includes all those phenomena based upon the fact that in an explosive mixture heated to a small degree the rate of reaction as the result of the infinitesimal change increases tremenduously. This leads to the study of the reaction preceding the actual explosion. The

slower processes of combustion are given attention covering the kinetics of the oxidation of lower aldehydes and of the lowest hydrocarbons such as acetylene, ethylene, ethane and propane. Regarding the complete explosive reaction attention is given to the products of reaction, temperatures reached, heat liberated and what radiation emanates.

Interest is centered in the mechanism of the progress of an explosion including the speed with which an explosion, started in one place, is propagated to the unburned mixture. It is shown that if the unburned gas is static there is the so-called ignition speed or normal speed of combustion, that the actual speed of combustion can by the aid of currents be much greater than the normal speed of combustion, and that if the speed of propagation of a combustion is very high a shock wave may occur and in extreme cases "detonation" occurs resulting from a combination of shock wave and chemical reaction. Attention is given to unpremixed gases and gases not sufficiently premixed. The latter part of the book is devoted to combustion in Otto engines and in the Diesel engine. In the back of the book there is a list of references, a name index and a subject index.

The book is a compilation of a great deal of scattered information coupled with the contribution of expert knowledge presented in a logical manner. It should find much use as a reference work, as well as a text.

R. H. OPPERMANN.

CHEMISTRY OF FAMILIAR THINGS, by Samuel Schmucker Sadtler. Eighth edition, revised. 310 pages, illustrations, 14 × 21 cms. Philadelphia, J. B. Lippincott Company, 1946. Price $4.00.

As the many new products of scientific knowledge enter into our everyday life, there is an increasing need for the well informed layman to be acquainted with the fundamentals from which they are produced as a matter of culture as well as a means of greater appreciation of them. Chemistry can furnish interesting subject matter along these lines for general consideration, although it has not appealed to very many people who were not themselves chemists. The method is through an introduction by a simple and understandable exposition of familiar things which surround us in our daily lives. This book is such a work.

Now in its eighth edition, revised, since 1915, this book undertakes to make the layman familiar with fundamental and accurate information sufficient for greater appreciation of the science and its products. The beginning is concerned with a description of fundamentals—easy and interesting, but this may be skipped if the reader is not in the mood. A large part of the book is devoted to the chemistry of such subjects as air, water, metals, rocks, sod, food, textiles, chemical evolution and physiological chemistry. The treatment is one of logical presentation in a simple style, always from the human interest standpoint. At times the story seems to drift somewhat from what is strictly chemistry into other subjects, for instance, biology and food values. Thus, there is a chapter on foods with vitamins which discusses oysters, their content, their action in the digestive system, and their effects. In like manner lobsters, fish, eggs, milk, etc., are treated.

Of a more chemical nature is the discussion of yeast and sugar involving fermentation leading into the production of wines. Also health through chemistry is in the same category, where it is pointed out that the living body is a laboratory in which an innumerable number of chemical changes are continually taking place, and just as in a highly organized manufacturing laboratory, when orders are not given or miscarried, processes will go wrong and much damage to the products and even to the plant may result. Later the monomers, polymers and elastomers are given attention. This portion of the book is not as appealing as others, but the silicious substances and glass is a subject quite interesting.

The fact that a book of this nature has had so many editions is indicative of its popularity and the present edition has the benefit of experience. Those who are interested in scientific matters in a general way but who do not wish to become deeply involved will find here an interesting, informative, descriptive work.

R. H. OPPERMANN.

CHARLES DARWIN AND THE VOYAGE OF THE BEAGLE. Edited with an introduction by NORA Barlow. 279 pages, illustrations, 14 × 21 cms. New York, The Philosophical Library, Inc., 1946. Price $3.75.

This name will always be principally associated with the theory of evolution. That Charles Darwin's scientific interest started early is of course known, but the details of his youthful adventure on the voyage of circumnavigation are not so well known. This volume deals with an early period of Darwin's life, in the year of 1831 when he set sail in the *Beagle* on an expedition for the purpose of making an accurate survey of the southern coast lines of the South American continent together with running a chain of chronometric readings round the world. During the 5 years of the voyage, Darwin pursued the evasive geological puzzles that met his inexperienced eyes and made vast collections of animals and plants. It created influences on his mind and character resulting in great changes.

The main content of this book is a series of 36 letters written by Darwin to his family and 24 little pocket books arranged and edited by Lady Barlow who is Charles Darwin's granddaughter and a notable Darwin scholar. The introduction furnishes a background for this work by describing the home where Darwin was reared, the family, his education, and events which led to the offer and acceptance of Darwin for the voyage of the *Beagle*. The second part of the book contains the letters, and the third part the note books. The study of the two parallel sets of manuscripts yields further light on Charles Darwin as he then was, emotionally sensitive and intellectually malleable. It is a fitting addition to the works on Darwin.

R. H. OPPERMANN.

RADIO'S CONQUEST OF SPACE, BY Donald McNicol. 374 pages, illustrations, 14 × 22 cms. New York, Murray Hill Books Inc., 1946. Price $4.00.

The time is drawing to a close when source material can be written about the beginning of radio: the pioneers are already few in number. Later other histories will be written, but what is then presented will perforce have to be based upon the early published matter. The author of this book was, in effect, a practicing radio engineer long before the term "radio engineer" was thought of. In 1900 he constructed the first experimental wireless transmitter and receiver in the Midwest and in 1907 organized the first amateur radio club in Salt Lake City. This book is written with the belief that care should be exercised, that the foundations of the science and industry be truthfully recorded.

The story is essentially non-technical in describing the technical developments from the time of Hertz's waves to the present day. Written in a general logical sequence it reviews the discoveries that contributed to the conception and development of radio signaling, and considers the experiments that led to the construction of induction coils and transformers in response to the need for very high voltages and high frequencies of alternating current. Space is then devoted to a review of various detectors discovered, invented, improved or otherwise acquired by the physicists, engineers, and amateur workers in early wireless signaling. Record is made of the examinations and conceptions of the structure of the wave and the nature of radiation, leading to the theory of the Kennelly-Heaviside layer. The evolution of the audion is traced, and the development of the vacuum tube related which led to more knowledge on electric wave propagation and the trend to shorter wave lengths. After brief reference to wire telephony, the story is concerned with improvements in receivers and transmitters and the expanding sphere of radio in peace time and in war time.

This is not a history, but a record of experimental results, discoveries, and developments presented in an interesting fashion easily understood by the layman.

R. H. OPPERMANN.

SCIENCE REMAKES OUR WORLD, by James Stokley. 318 pages, plates, 15 × 23 cms. New York; Ives Washburn, 1946. Price $3.50.

The present volume is a revised edition of the first copy which was printed in 1942 and followed by a second and third printing in 1943 and 1944. More material has been added to

this present edition. Chapter XIX has been enlarged and Chapter XXI has been added. The material that appears in these two chapters was at the time of the war secret and confidential but now the author has been able to write about the atomic bomb and the method arrived at through research in developing such a war weapon.

Many excellent reviews have been made of the entire book as it formerly appeared and the present revised edition deserves the same praise.

The author is well known through the continuous amount of material that he has written over a long period. He is well versed in all the different fields of science and through this he has been able to write this book in a concise and exact manner.

W. A. R. PERTUCH.

BUTALASTIC POLYMERS: THEIR PREPARATION AND APPLICATIONS, by Frederick Marchionnia. 642 pages, plate, 15 × 23 cms. New York, Reinhold Publishing Corporation, 1946. Price $8.50.

The word "butalastics" is adopted in this book to describe and discuss the elastic or plastic polymers of a butadiene compound with or without other compounds polymerizable with it. They are the true synthetic rubber-like products. The term is the result of attempts to satisfy a need for a scientific name, the meaning of which is all-inclusive. With the name is presented a scheme of classification for naming all known butalastic polymers. This has seventeen classes.

The plan of the book is to give the history of the development of the butalastic products; the source and production of the raw materials used for forming them; the mechanism of polymerization; the methods of polymerization and copolymerization or interpolymerization of the various compounds; the compounding practice; industrial application and cure of the polymers; the properties and testing of the products; the reclaiming of the butalastics waste; and derivatives of the butalastics. The subject is treated at great length in a rigorous fashion containing a fund of information in text form as well as tables, curves and configurations. Innumerable references are made to a wide field of the literature including patents of the United States and foreign countries. In the back of the book is a United States Patent list in numerical order, an author index, an index of catalysts of polymerization, and a very complete index of subjects including separation of polymers having two different monomers or one polymer with a monomer, and polymers with three or more monomers.

The work is an exhaustive treatment making available the extensive data on the production, properties and uses of the known butalastics which are acquiring more and greater importance in national and international economics, relations, and social intercourse. It is a useful guide and stimulus to future investigators.

R. H. OPPERMANN.

THE AMERICAN ANNUAL OF PHOTOGRAPHY 1947, edited by Frank R. Fraprie and Franklin I. Jordan. 208 pages, 19 × 25 cms. Boston, American Photographic Publishing Co., 1946.

This well known photographic annual which is now appearing in its 61st volume was first published in the year 1887 under the title of American Annual of Photography and Photographic Times Almanac.

The publishers have followed the same idea of including many fine reproductions of photographs through all these years. In this present volume the illustrations form a well chosen collection of photographs of the year. Both the text and illustrations are well arranged and the size of the illustrations helps to bring out the details in the photographs. In the text there are articles both technical and instructive. Some of these articles deal with the electron microscope, developers, photographing historic landmarks, color photography, pictorial illustrations and gum prints. The book also includes a list of exhibition records for the past three years of photographers who had pictorial prints accepted by two salons within the past exhibition years and a list of amateur photographic organizations both in the United States and abroad.

W. P.

ADVANCED MECHANICS OF MATERIALS by Glenn Murphy. First Edition, 307 pages, illustrations, 15 × 24 cms. New York, McGraw-Hill Book Company, 1946. Price $4.00.

Designed for an advanced undergraduate or graduate course in strength of materials, this book deals with the determination of the stresses in some of the more commonly used complex members, in comparison with elementary mechanics of materials which includes the study of axially loaded members, thin walled pressure vessels, straight shafts and beams at constant cross section and columns. At the outset there is a general discussion of the relationship of materials to design including the significance of stress and failure, types of failure and of loading, factor of safety, stiffness and other properties. This is followed by sections devoted to stresses and strains at a point, stress-strain relationships, theories of failure and axial loading.

An example of the emphasis throughout the book can be seen in the treatment of cylinders, spheres and discs under radially symmetrical loads that the stress is dependent upon each of the three items, statics, geometry, and the properties of the material. This is again carried through in the section devoted to torsion.

The coverage is of the force system acting at a given cross section of a member which may be resolved into a force normal to the section and acting through the centroid producing axial loading, a force lying in the plane of the cross section producing cross shear, a couple lying in the plane of the cross section producing torsion and a couple lying in a plane normal to the plane of the cross section producing flexure or bending. In the latter part there is a section devoted to a preview of photoelastic analysis which precedes a treatment on compressive loads and buckling, the evaluation of stresses in the vicinity of concentrated loads.

This presentation is rigorous but plain. The author believes that impressions are better made by utilization of knowledge in the solution of problems. Accordingly, many exercises are given at the ends of chapters. The answers are given in the back of the book. There is an author and subject index useful for reference.

R. H. OPPERMANN.

PUBLICATIONS RECEIVED.

A New Notation and Enumeration System for Organic Compounds, by G. Malcolm Dyson. 63 pages, tables, 15 × 24 cms. New York, Longmans, Green and Co., Inc., 1947. Price $1.75 (paper).

The Poetry of Mathematics and Other Essays, by David Eugene Smith. Second Printing. 90 pages, 13 × 18 cms. New York, Scripta Mathematica, 1947.

German Research in World War II, by Leslie E. Simon. 218 pages, drawings and illustrations, 15 × 24 cms. New York: John Wiley & Sons, Inc., London: Chapman & Hall, Limited, 1947. Price $4.00.

Textile Fibers, by J. M. Matthews and H. R. Mauersberger. Fifth Edition. 1133 pages, tables and illustrations, 15 × 24 cms. New York; John Wiley & Sons, Inc., London; Chapman & Hall, Limited, 1947. Price $12.50.

The Travels of William Bartram, edited by Mark Van Doren. 414 pages, 14 × 21 cms. New York, Dover Publications, 1928. Price $3.75.

Portraits of Eminent Mathematicians, With Brief Biographical Sketches by David Eugene Smith. Portfolios 25 × 34 cms. New York, Scripta Mathematica, 1946. Price $5.00.

Low-Pressure Laminating of Plastics, by J. S. Hicks. 162 pages, drawings and illustrations. 16 × 23 cms. New York, Reinhold Publishing Corporation, 1947. Price $4.50.

The Chemistry and Technology of Plastics, by Raymond Nauth. 522 pages, drawings, 16 × 23 cms, and illustrations. New York, Reinhold Publishing Corporation, 1947. Price $9.50.

CURRENT TOPICS.

Electrons Provide Means of Analyzing Metal Surfaces. (*Machinery*, Vol. 53, No. 5.)—The use of a stream of electrons to study the atomic structure of metals and gases and to identify invisible layers of films of foreign materials on metal surfaces has been developed into a practical process in the chemistry laboratoires of the University of Michigan. This provides an "electron diffraction" method of analyzing metal surfaces which permits a wide variety of practical applications in the fields of welding, lubricating, electroplating, and painting. A machine based on an experimental model developed by Professor Lawrence O. Brockway is now on the market for this purpose.

This machine produces a stream of 40,000-volt electrons in a vacuum chamber. In the center of the chamber is a specimen holder on which the piece of metal to be studied is placed. This metal interrupts the electron stream so that the electrons bend or break off into new paths. As they do so, they expose a photographic plate that is inserted at the end of the chamber.

Every metal breaks up the electron stream in a distinct pattern so that photographs resulting from this process are different for each metal. It is, therefore, possible to identify unknown substances in the electron stream by comparing their "diffraction patterns" or photographs with those of substances already identified.

R. H. OPPERMANN.

Flights Confirm Artificial Snow Production.—Experiments in the artificial production of snow by "seeding" with dry ice a cloud of water droplets cooled below the freezing point have fully confirmed earlier tests of the method, it was revealed at a meeting of the American Meteorological Society in Boston, when Dr. Irving Langmuir, associate director of the General Electric Research Laboratory, and Vincent J. Schaefer, his associate, described the work they had done since Mr. Schaefer's original flight.

If it were completely utilized, Dr. Langmuir stated, a single one-quarter inch pellet of dry ice, which is solid carbon dioxide, could produce about 300,000 tons of snow. However, this cannot be achieved because an ice particle less than a millionth of a centimeter diameter will evaporate even if water is present. Thus, only the bigger particles survive. The total number of nuclei produced, according to his calculations, is of astronomical magnitude. A single particle of a quarter of an inch diameter will produce around 10,000,000,000,000,000 nuclei!

Mr. Schaefer said that while his latest experiments demonstrate conclusively that when conditions are right it is possible to start a snowfall by seeding with dry ice, they show that many factors must be considered.

"In order to produce considerable amounts of snow on the ground a continuous flow of moisture into the region is required," he told the meteorologists. "Where there are only a few isolated clouds in the sky, the method will tend to dissipate them rather than to form more."

He explained that if a cloud two miles thick were entirely precipitated, there only would be about an inch of snow on the ground.

Mr. Schaefer's second flight, made November 22 from Schenectady toward the Massachusetts border, caused the billowing tops of cumulus clouds through which he flew to subside when the dry ice was dropped. On his return he flew under this region, and through heavy snow, falling from the clouds. A vapor trail behind the plane was produced by dropping the material as he flew along under a cloud near Schenectady.

On his third flight, November 29, the clouds were on the edge of a region of dry air, and too much dry ice was dropped, which produced too many nuclei in the cloud. The result was that no snow fell, and it looked the same as before the seeding run. However, the optical effects when he observed the sun through the cloud showed that tiny droplets of water had been converted to ice crystals. These were so small that they did not drop, since there was no additional supply of moisture.

He also explained how the method may be used in connection with winter fogs. These are generally supercooled like the clouds from which snow may be precipitated. That is, though they are at a temperature well below the normal freezing point of water, the droplets are still liquid. He made one experiment when such a fog covered the river flats near his home. Waving a wire basket full of dry ice around his head as he walked through the fog, he changed its character completely and eventually left a clear space where he had walked.

Later, on Mt. Washington, his associate, R. S. Johannson, tried some chilled carbon dioxide gas on an icing cloud. At first it got thicker but then as the crystals grew they settled on the ground as a barely visible layer. So much moisture was taken out of the air at this point that no further fog could form for some time. This clear area did not remain fixed. The entire cloud was moving so the open space moved along with it.

Mr. Schaefer told the meteorologists that his war work on icing conditions had led to a consideration of the mechanism of ice crystal formation. He found, he said, "that a cloud may consist of liquid water droplets down to −31°Fahrenheit. Below this point they freeze spontaneously. The pellets of dry ice, at −78° Fahrenheit, as they pass through moist air, create in their immediate vicinity submicroscopic condensation nuclei. These, in their initial stage, are probably spherical droplets of supercooled water, but they soon freeze and develop as ice crystals."

Mr. Schaefer told of another experiment with a cold chamber, kept slightly below −31° Fahrenheit, with a continual supply of moisture. Though the chamber was tightly sealed there was a continuous snowfall for two weeks. This shows, he said, that foreign nuclei are not necessary to produce ice crystal formation at this low temperature.

Negotiations are at present being concluded with the United States Army and further experimentation in this project will probably be done in co-operation with the Evans Signal Corps Laboratory at Camp Evans, New Jersey. The U. S. Army and Navy Air Forces will also participate.

R. H. O.

Welfare of Generations Depends on Metallurgists. (*Blast Furnace and Steel Plant*, Vol. 34, No. 12.)—Metallurgists are custodians of materials on which the welfare of countless generations must depend, stated Walter S. Tower, president of the American Iron and Steel Institute, before members of the American Society for Metals at their annual dinner. Mr. Tower said that every advance in the science of metallurgy tends to make metals more serviceable, to enhance their economic value, and to prolong the life of existing natural resources.

Mr. Tower stated, "The metallurgist has dug deep into his bag of tricks and come up with industrial applications of elements which were, until a few years ago, largely laboratory items. Our whole manner of living would be far different without the wide range of qualities of the numerous alloys which the metallurgist has made available. Many of these metallurgical marvels have been designed specifically for the benefit of the distaff side of the house, such as the gleaming stainless steel sink, the enameled gas range, stainless or clad metal pots and pans, satin-like silver and keen carving tools.

"The wider and more attractive horizons to lure the venturesome metallurgist now seem to lie in the direction of preparing metals for use, since the basic knowledge of separating metals from ores has been pretty well solved and great strides have likewise been made in the metals and economies of refining metals."

Commenting on some belief that the use of ferrous metals (steel iron and their alloys) may be displaced in the not distant future because of desirable qualities of the light metals, aluminum and magnesium, Mr. Tower declared, "It does not seem likely that future progress of metallurgical skills will bring any considerable displacement of ferrous metals by the non-ferrous. It is likely, and perhaps desirable, that in some uses other metals may take the place which some kind of steel has heretofore occupied. However, as we know metals, it seems clear that the broad foundation of industrial use of metallic products must continue indefinitely to rest on the ferrous family. That conclusion is supported by the facts in regard to potential supply, prospective cost of production, price to the user and versatility in use.

"In none of those four respects has the ferrous family any serious rival. What the metallurgist has done and can still hope to do in preparation of ferrous metals for use has no parallel elsewhere.

"In every consideration of the use of metals, it is important to recognize that we are dealing with exhaustible resources. Every ton of metal extracted, refined and put to use is an irreplaceable subtraction from the aggregate which this earth offers. Some metals, once used, can be recovered from scrap, and in secondary form contribute substantially and repeatedly to the needs of industry. But existing resources cannot be increased by any magic of metallurgists. Synthetic metals in quantities are beyond the imagining of fiction. You metallurgists, therefore, are custodians of materials on which the welfare of countless generations must depend."

R. H. OPPERMANN.

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